Yuga Cycles and Cosmic Energy Flow:
A Dynamical Analysis of the Solar System's Libration in the Milky Way

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The Yuga is a cosmic chronology in Hindu thought that divides the history of the Solar System into four epochs—Satya Yuga, Treta Yuga, Dvapara Yuga, and Kali Yuga—based on the periodic libration of the Solar System within the Milky Way Galaxy. This framework arises from the premise that the radiative energy received on Earth from galactic stars oscillates as a consequence of this libration. The creation and decline of the planetary network of life are thus driven by the influx of cosmic energy, with structural transformations shaping its diverse manifestations. In this paper, we undertake a scientific examination of Yuga theory by physically analyzing the orbital dynamics of the Solar System within the Galaxy. We show that the anisotropic structure of the Milky Way governs the periodicity of libration and thereby modulates the Yuga cycles. Furthermore, we investigate the relationship between the stellar radiative energy observed from Earth and the dissipation of energy within the planet, and we argue that Landauer's principle provides a foundation for understanding the diversity of life's informational complexity and clarifies the defining characteristics of each epoch in the Yuga cycle.

Introduction

The Earth, as a member of the Solar System, depends primarily on radiative energy from the Sun to sustain its activity. The planet's daily rotation produces the alternation of day and night, while its annual revolution generates the reversal of seasons between the Northern and Southern Hemispheres. These variations occur on time scales short enough to be perceptible to life on Earth. However, when viewed against the 225–250 million-year galactic period of the Solar System's orbit within the Milky Way (the so-called "galactic year"), the solar radiative input may be regarded as effectively constant[1].

The Solar System, belonging to the Milky Way Galaxy, revolves around a supermassive black hole located near Sgr$A^\star$, which is considered to be its approximate galactic nucleus. Its orbital path is thought to be elliptical. The Milky Way is classified as a barred spiral galaxy[2], and the gravitational forces arising from its complex morphology induce a phenomenon known as libration, in which the Solar System oscillates in a direction perpendicular to the plane of its elliptical orbit[2].

In the Western tradition, historical time is conventionally reckoned from the death of Christ, and for many centuries—from the 2nd until the mid-15th century—the Ptolemaic geocentric model of Egypt remained widely accepted. In the 15th century, however, Copernicus, Kepler, and Galileo advanced the heliocentric theory, which for the first time led to a scientific understanding of the Earth's orbital motion and introduced the possibility that the Solar System itself might revolve around a larger center. Yet the notion that the Sun completes a revolution around the Galaxy in one galactic year (approximately 220 million years) was not established until the early 20th century, when Shapley and others elucidated the scale and structure of the Milky Way.

By contrast, in the Eastern tradition, concepts of time and of solar motion were known far earlier. In Hinduism, for example, the Vedas (ca. 1500 BCE) already articulated ideas of time and periodicity, and the Yuga (the "divine year") emerged as a recognized temporal unit. The Mahābhārata (ca. 400 BCE–200 CE) clearly described the four Yuga ages and helped disseminate the concept. The Purāṇas (ca. 4th–11th centuries CE) offered detailed explanations of the Yuga epochs and advanced a systematic cosmology of creation and destruction. Similarly, early Buddhist scriptures of around the 5th century BCE proposed cyclical notions of time. In the Buddha's teaching, the kalpa served as a unit of cosmic time, and in the Abhidharma (ca. 4th–5th centuries CE) Buddhist cosmology was further systematized, elaborating such concepts as the kalpa, the mahākalpa, and the Kalachakra of Tibetan Buddhism.

The Yuga theory, originating in Hindu Purāṇic literature, posits a recurring creation and dissolution of civilizations on the galactic-year timescale of the Solar System[4],[5]. One complete Yuga cycle spans 4,320,000 years and is subdivided into four successive ages of 12,000 divine years each—Krita (Satya) Yuga, Treta Yuga, Dvapara Yuga, and Kali Yuga—as follows:

Satya Yuga 1,728,000 years (4,800 divine years)

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[2]The libration of the Moon has been widely investigated[3].

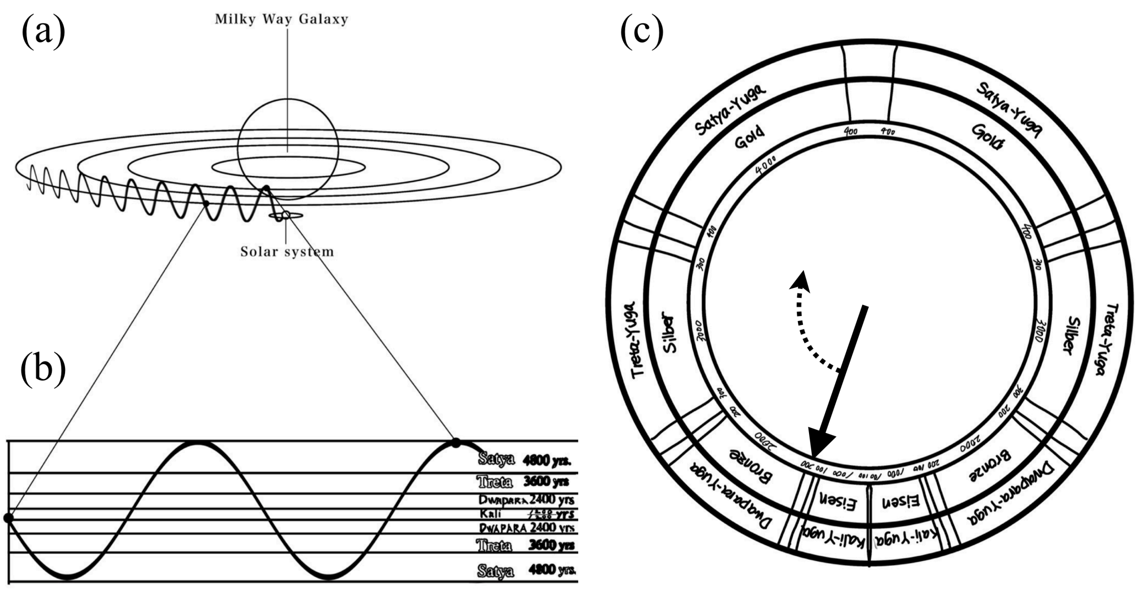

Figure 1: Solar-system libration and the Yuga cycle: (a) Libration of the Solar System as it orbits the Milky Way Galaxy. (b) Relationship between libration and the Yuga cycle as proposed in Yuga theory. (c) The Yuga clock, depicted with left–right symmetry.

Also called the "Golden Age," this era was said to be governed entirely by truth and virtue. Humanity lived in purity and moral harmony, with an average lifespan of 400 years and a stature of 21 cubits ($\approx$ 9.5 m).

Treta Yuga 1,296,000 years (3,600 divine years)

Known as the "Silver Age," it emphasized religious rituals and sacrifices. Human thought was characterized by a three-to-one ratio of virtue to sin. Although moral decline began, society remained relatively stable. Average lifespan was about 300 years, and height about 14 cubits ($\approx$ 6.3 m).

Dvapara Yuga 864,000 years (2,400 divine years)

Referred to as the "Bronze Age," this epoch was marked by conflict and a balance of good and evil. The average human lifespan was about 200 years, with a height of 7 cubits ($\approx$ 3.2 m).

Kali Yuga 432,000 years (1,200 divine years)

Identified as the "Iron Age," this is the present era[3]. It is described as dominated by three-quarters sin and one-quarter virtue, characterized by poverty, hostility, and instability, with an average human lifespan of about 100 years and a stature of 3.5 cubits ($\approx$ 1.6 m).

The relationship between solar-system libration and the Yuga cycle is illustrated in Figure 1. The periodicity of the Solar System's orbital libration is closely tied to the Yuga sequence, and it is often suggested that we now stand near the boundary between the Kali and Dvapara Yugas.

In this paper, we propose a model that analyzes the Solar System's orbital motion around the galactic center and its dependence on radiative heat originating from the stars of the Milky Way. From the energy obtained through such radiation, we estimate the corresponding energy dissipation using current observational data. According to Landauer's principle, this dissipation provides a measure of the accessible information concerning the Earth, the biosphere, and the technological sphere[6]. This framework offers insight into the Yuga cycle and the recurring rise and decline of human cultures on Earth, together with the distinctive characteristics of each epoch.

Model

The Milky Way Galaxy exhibits a complex morphology, featuring not only a central bulge but also ripples and spiral arms forming rotating vortices. For simplicity, we consider an approximation that does not attempt to model this structure in full detail; instead, the Galaxy is treated as a massive triaxial ellipsoid with uniformly distributed mass (see Figure 2).

Because the Solar System lies sufficiently far from the supermassive black hole at the Galactic center, the effects of general relativity can be neglected. Furthermore, the Solar System may be treated here as a point mass m. According to Newtonian gravitation, the gravitational potential Φ of the Milky Way is a solution of the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho. \tag{1}$$

[3] Some interpretations, citing modern developments such as electricity and nuclear energy, argue that we are still in Dvapara Yuga.

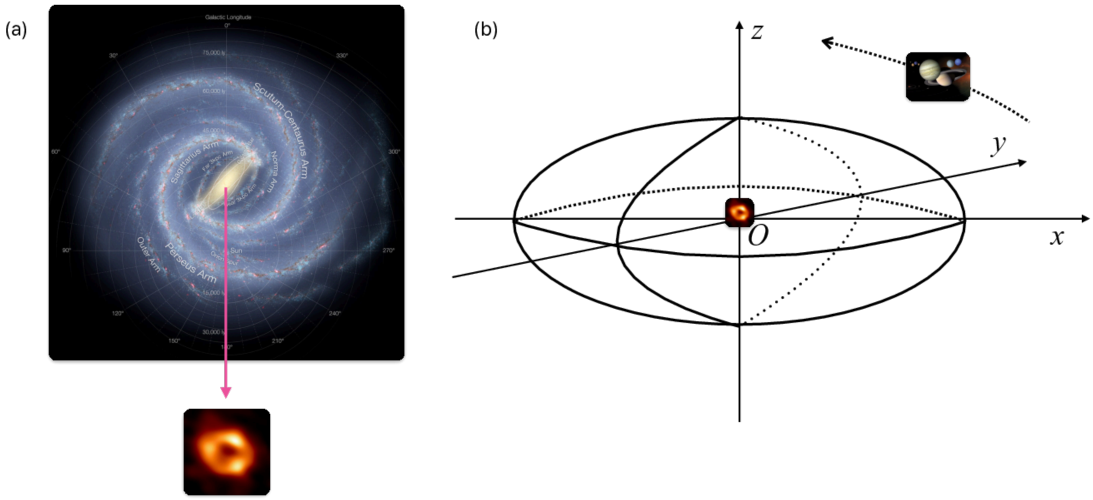

Figure 2: Solar System orbiting the Milky Way and its approximate model: (a) The Milky Way possesses a bar-shaped bulge in which a large concentration of stars resides. At its center, a supermassive black hole, Sgr$A^\star$, has been observed. The Solar System revolves around this bulge. (b) In this study, the Galactic mass distribution is approximated as a triaxial ellipsoid—taking axis lengths a, b, and c along the x, y, and z directions, respectively—to compute the orbital motion of the Solar System.

Hence, taking the origin at the Galactic center and denoting the position vector by $\boldsymbol{r}$, the gravitational potential exerted on the Solar System by the Milky Way is

$$\Phi(\boldsymbol{r}) = -\int_V G\frac{\rho}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}', \tag{2}$$

where the integration extends over the entire Galaxy, here approximated as a triaxial ellipsoid. Expanding this in multipoles yields

$$\Phi(\boldsymbol{r}) = -G\left(\frac{M}{r} + \frac{1}{2}Q^{ij}\frac{r^i r^j}{r^3} + \cdots\right) \tag{3}$$

(see Appendix, Equation 131) . Because of the assumed uniform mass distribution, all odd-order moments vanish.

If the Milky Way rotates with a constant angular velocity $\boldsymbol{\Omega}$, the velocity of the Solar System $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{v} = \begin{pmatrix} \dot{x} - \Omega y \\ \dot{y} + \Omega x \\ \dot{z} \end{pmatrix}_{\mathrm{G}} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r(\dot{\varphi} + \Omega)\cos(\theta) \end{pmatrix}_{\mathrm{polar}} \tag{4}$$

(see Appendix, Equation 85 and Equation 91).

To derive the equations of motion, we define the Lagrangian

$$\begin{aligned} L(\boldsymbol{r}, \dot{\boldsymbol{r}}) &\equiv T(\boldsymbol{v}(\dot{\boldsymbol{r}}, \boldsymbol{r})) - m\Phi(\boldsymbol{r}), \\ T(\boldsymbol{v}(\dot{\boldsymbol{r}}, \boldsymbol{r})) &\equiv \frac{1}{2}mv^2. \end{aligned} \tag{5}$$

Applying the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} - F^i \approx 0 \tag{6}$$

(see Appendix, Equation 99), and neglecting radiative stresses from nearby stars and collisions with extragalactic bodies, we obtain the equation of motion

$$m\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}_{\mathrm{G}} = 2m\Omega\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix}_{\mathrm{G}} + m\Omega^2\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}_{\mathrm{G}} + \boldsymbol{F}, \tag{7}$$

or, in polar coordinates,

$$\begin{aligned} &m\begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\cos^2(\theta) \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} + r\dot{\varphi}^2\cos(\theta)\sin(\theta) \\ (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos(\theta) - 2r\dot{\theta}\dot{\varphi}\sin(\theta) \end{pmatrix}_{\mathrm{polar}} \\ &= 2m\Omega\begin{pmatrix} r\dot{\varphi}\cos^2(\theta) \\ -r\dot{\varphi}\cos(\theta)\sin(\theta) \\ -\dot{r}\cos(\theta) + r\dot{\theta}\sin(\theta) \end{pmatrix}_{\mathrm{polar}} \\ &+m\Omega^2 r\cos(\theta)\begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{pmatrix}_{\mathrm{polar}} + \boldsymbol{F} \end{aligned} \tag{8}$$

(see Appendix, Equation 106 and Equation 113). The terms on the right-hand side correspond respectively to the Coriolis force, the centrifugal force, and gravity. The gravitational force is

$$\begin{aligned} \boldsymbol{F} &= -m\boldsymbol{\nabla}\Phi(\boldsymbol{r}) \\ &= \boldsymbol{F}^{(m)} + \boldsymbol{F}^{(q)} + \cdots, \end{aligned} \tag{9}$$

where $\boldsymbol{F}^{(m)}$, $\boldsymbol{F}^{(q)}$ denote the monopole and quadrupole contributions, respectively (see Appendix, Equation 139 and Equation 155). Specifically, the quadrupole forces is

$$\boldsymbol{F}^{(q)} = m\frac{G}{2r^4}\boldsymbol{f}, \tag{10}$$

where,

$$\begin{aligned} f^r &= -\frac{3}{5}M(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\ &\quad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\ &\quad +q^{zz}\sin^2(\theta)), \\ f^\theta &= \frac{1}{5}M\sin(2\theta)(-q^{xx}\cos^2(\varphi) \\ &\quad -q^{yy}\sin^2(\varphi)+q^{zz}), \\ f^\varphi &= \frac{2}{5}M(-q^{xx}\cos(\theta)\cos(\varphi)\sin(\varphi) \\ &\quad +q^{yy}\sin(\theta)\cos(\varphi)). \end{aligned} \tag{11}$$

Solutions for several simple cases, such as the Kepler problem, have already been clarified through the analyses in orbital mechanics [7] (see Appendix). Our calculation considers the effects of the quadrupole moment within a model that accounts for the anisotropy of the Milky Way galaxy. This anisotropic constraint force induces a novel spiral motion of the Solar System.

Results and Discussion

The dynamical model we propose provides a spiral orbital solution of the Solar System around the Galaxy, arising from the galactic anisotropy. Through perturbative calculations based on this model, we obtain a quantitative assessment of the Yuga cycle. By rearranging the equations of motion of the model, Equation 8 (also see Equation 113), into component form and rewriting them using conserved quantities such as angular momentum (see Appendix, Equation 227, Equation 229, and Equation 230), we obtain the following:

- r-direction

$$\begin{aligned} \ddot{r} &= \frac{1}{r^3}\left(h_\varphi^2 + h_\theta^2\right) \\ &\quad -G\frac{M}{r^2} \\ &\quad -GM\frac{3}{10r^4}(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\ &\qquad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\ &\qquad +q^{zz}\sin^2(\theta)) \end{aligned} \tag{12}$$

where

$$h_\varphi \equiv r^2(\dot{\varphi}+\Omega)\cos(\theta), \quad h_\theta \equiv r^2\dot{\theta} \tag{13}$$

- θ-direction

$$\begin{aligned} \dot{h_\theta} &= -\frac{h_\varphi^2}{r^2}\tan(\theta) \\ &\quad -G\frac{M}{10r^3}(q^{xx}-q^{yy})\sin^2(\varphi) \end{aligned} \tag{14}$$

- φ-direction

$$\begin{aligned} \dot{h_\varphi} &= \frac{h_\theta}{r^2}h_\varphi\tan(\theta) \\ &\quad -G\frac{M}{5r^3}\cos(\varphi)(q^{xx}\cos(\theta)\sin(\varphi)-q^{yy}\sin(\theta)) \end{aligned} \tag{15}$$

Building upon this framework, we discuss how approximate calculations in the r- and θ-directions enable us to derive quantitative expressions for both the Yuga cycle and the ratio of radiative energy.

Quantitative derivation of yuga cycle by perturbation analysis. Let us next examine the oscillatory behavior under the assumption that θ is sufficiently small and that variations in r and φ evolve much more slowly compared to changes in θ. From the equation of motion in the θ direction, Equation 229, we may approximate

$$\dot{h}_\theta \approx -\left(\frac{h_\varphi}{r^2}\right)^2 h_\theta + \text{const.}. \tag{16}$$

Thus, the oscillation frequency in the θ direction, ω_θ, can be approximated as

$$\omega_\theta \approx \frac{h_\varphi}{r^2} \approx \frac{h}{r^2} = \dot{\varphi}+\Omega. \tag{17}$$

That is, the oscillation period T_θ in the elevational (θ) direction of the solar system is given by

$$T_\theta \approx \frac{1}{\omega_\theta} \approx \frac{r^2}{h} = (\dot{\varphi}+\Omega)^{-1}. \tag{18}$$

The orbital velocity of the solar system on the galactic plane,

$$v_\varphi \approx r(\dot{\varphi}+\Omega), \tag{19}$$

is observationally estimated as $v_\varphi \approx 240\,\mathrm{km\,s^{-1}}$, with $r \approx 8.0\,\mathrm{kpc} \approx 2.4 \times 10^{17}\,\mathrm{km}$ [8], [9].

From these values, we obtain

$$T_\theta \approx \frac{r}{v_\varphi} \approx 10^{15}\,\mathrm{s} = 3.2 \times 10^7\,\mathrm{yr}\ . \tag{20}$$

The Yuga cycle period predicted by Yuga theory (see Figure 1) is

$$\left(\frac{0.1728}{2}+0.1296+0.0864+\frac{0.0432}{2}\right)\times 10^7\times 4 \\ = 1.296\times 10^7\ \mathrm{yr}, \tag{21}$$

which demonstrates that the approximate oscillation period derived here is in reasonable agreement with the Yuga cycle.

Energy radiation from Milky way. Finally, we consider the energy radiation from the Galaxy to the Solar System, which plays a central role in the theory of the Yuga cycle. In this consideration, we assume that the radiative energy is distributed isotropically in space; that is, the radiated power depends inversely on the square of the distance between the Galaxy and the Solar System:

$$P\propto\frac{1}{r^2}. \tag{22}$$

Under this assumption, we investigate the minimum P_{min} and maximum P_{max} values of the radiated energy. From Equation 22, the condition of minimal radiation $P=P_{\mathrm{min}}$ corresponds to the maximal distance $r=r_{\mathrm{max}}$, whereas maximal radiation $P=P_{\mathrm{max}}$ corresponds to the minimal distance $r=r_{\mathrm{min}}$. Once the relation between r_{max} and r_{min} is determined, the corresponding energy ratio can be evaluated.

We next perform approximations related to Equation 12. As an approximation method, we describe a small spiral deviation from a Keplerian orbit. Furthermore, we assume that the Galactic major axis coincides with the x-axis, and the intermediate axis is taken as the b-axis ($a>c>b$). Under this assumption, one can expect $r\approx r_{\mathrm{min}}$ in the vicinity of $\varphi\approx 0,\pm\pi$, and $r\approx r_{\mathrm{max}}$ near $\varphi\approx\pm\frac{\pi}{2}$. If the Keplerian orbit is restricted to the xy-plane, these extrema in distance appear along $\theta\approx 0$. At such points of maximum or minimum distance, the radial variation halts, implying $\ddot{r}=0$. Consequently, we obtain the following relations:

$$0=r_{\mathrm{min}}(\dot{\varphi}+\Omega)-G\frac{M}{r_{\mathrm{min}}^2}-GM\frac{3}{10r_{\mathrm{min}}^4}q^{xx}, \tag{23}$$

and,

$$0=r_{\mathrm{max}}(\dot{\varphi}+\Omega)-G\frac{M}{r_{\mathrm{max}}^2}-GM\frac{3}{10r_{\mathrm{max}}^4}q^{yy}, \tag{24}$$

where $\dot{\varphi}$ is approximated as nearly constant.

By rearranging these expressions, we derive the relation

$$\left(\frac{r_{\mathrm{max}}}{r_{\mathrm{min}}}\right)^2=\frac{q^{xx}}{q^{yy}}\frac{A\left(\frac{r_{\mathrm{max}}}{r_{\mathrm{min}}}\right)^3-\frac{1}{r_{\mathrm{min}}^3}}{A-\frac{1}{r_{\mathrm{min}}^3}}, \tag{25}$$

with $A\equiv\frac{\dot{\varphi}+\Omega}{GM}$.

Taking the characteristic length scale as $L_0\equiv\left(\frac{GM}{\dot{\varphi}+\Omega}\right)^{\frac{1}{3}}=A^{-\frac{1}{3}}$, and assuming $L_0=1\gg r_{\mathrm{min}}$, i.e., $\frac{1}{L_0^3}\ll\frac{1}{r_{\mathrm{min}}}$ or equivalently $\frac{1}{Ar_{\mathrm{min}}^3}\ll 1$, we find

$$\frac{r_{\mathrm{max}}}{r_{\mathrm{min}}}\approx\frac{q^{yy}}{q^{xx}}=\frac{2b^2-a^2-c^2}{2a^2-b^2-c^2}. \tag{26}$$

Therefore, the ratio of maximal to minimal radiated energy can also be expressed in terms of the Galactic anisotropy as

$$\frac{P_{\mathrm{max}}}{P_{\mathrm{min}}}\approx\left(\frac{r_{\mathrm{min}}}{r_{\mathrm{max}}}\right)^2\approx\left(\frac{2a^2-b^2-c^2}{2b^2-a^2-c^2}\right)^2 \tag{27}$$

From the bulge-shape ratios of the Milky Way Galaxy [10],

$$a:b:c=1.00:0.43:0.40, \tag{28}$$

we can estimate

$$\frac{P_{\mathrm{max}}}{P_{\mathrm{min}}}\approx\left(\frac{2-0.43^2-0.40^2}{2\cdot 0.43^2-1^2-0.40^2}\right)^2\approx 4.387. \tag{29}$$

Thus, the radiative energy ratio is estimated to be about 4.

Of this radiated energy P, a fraction can be converted—though with some fluctuations—into usable energy for humanity through technological innovations such as solar panels and other clean-energy systems based on Landauer's principle [6]. This suggests that the usable energy ratio available to humanity may vary by approximately a factor of four due to the Yuga cycle.

Conclusion

Our model successfully incorporates the anisotropy of the Galactic bulge. By accounting for this anisotropy, we quantitatively reproduce the Yuga cycle and demonstrate that the radiative energy from the Galaxy to the Solar System varies by a factor of approximately four. Such a fourfold variation in the usable energy ratio implies significant fluctuations during the Yuga epochs.

Moreover, relativistic corrections, such as those considered in post-Newtonian approximations [11], represent a potential avenue for future refinement. In addition, more rigorous analyses based on numerical simulations are also expected as future prospects.

Appendix

Coordinate transformation. In this study, we perform a transformation into a polar coordinate system whose origin is set at the Galactic center. To carry out these

$$K(\boldsymbol{\Omega}) = \begin{pmatrix} w_x^2(1-\cos(wt))+\cos(wt) & w_x w_y(1-\cos(wt))-w_z\sin(wt) & w_x w_z(1-\cos(wt))+w_y\sin(wt) \\ w_x w_y(1-\cos(wt))+w_z\sin(wt) & w_y^2(1-\cos(wt))+\cos(wt) & w_y w_z(1-\cos(wt))-w_x\sin(wt) \\ w_x w_z(1-\cos(wt))-w_y\sin(wt) & w_y w_z(1-\cos(wt))+w_x\sin(wt) & w_z^2(1-\cos(wt))+\cos(wt) \end{pmatrix} \quad (36)$$

where $w \equiv \Omega$, and $(w_x, w_y, w_z) \equiv \left(\frac{\Omega^X}{\Omega}, \frac{\Omega^Y}{\Omega}, \frac{\Omega^Z}{\Omega}\right)$

transformations correctly, we first organize the Galactic frame and its representations in both Cartesian and polar coordinates.

3D orthogonal coordinates

For the laboratory frame (world coordinate system), we define the basis vectors of the three-dimensional Cartesian system (X, Y, Z) as

$$\boldsymbol{n}_X \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{n}_Y \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{n}_Z \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (30)$$

These basis vectors form a right-handed system, so their cross products are defined as

$$\begin{aligned} \boldsymbol{n}_X \times \boldsymbol{n}_Y &= -\boldsymbol{n}_Y \times \boldsymbol{n}_X = \boldsymbol{n}_Z, \\ \boldsymbol{n}_Y \times \boldsymbol{n}_Z &= -\boldsymbol{n}_Z \times \boldsymbol{n}_Y = \boldsymbol{n}_X, \\ \boldsymbol{n}_Z \times \boldsymbol{n}_X &= -\boldsymbol{n}_X \times \boldsymbol{n}_Z = \boldsymbol{n}_Y. \end{aligned} \quad (31)$$

The cross product of any two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is expressed as

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} &= (A^X\boldsymbol{n}_X + A^Y\boldsymbol{n}_Y + A^Z\boldsymbol{n}_Z) \\ &\quad \times (B^X\boldsymbol{n}_X + B^Y\boldsymbol{n}_Y + B^Z\boldsymbol{n}_Z) \\ &= (A^Y B^Z - A^Z B^Y)\boldsymbol{n}_X \\ &\quad +(A^Z B^X - A^X B^Z)\boldsymbol{n}_Y \\ &\quad +(A^X B^Y - A^Y B^Z)\boldsymbol{n}_Z. \end{aligned} \quad (32)$$

Galaxy frame

Here, we treat the Milky Way Galaxy as a triaxial ellipsoid and define the x, y, and z axes along its principal directions with lengths (a, b, c). This is a non-inertial coordinate system that rotates with the Galactic angular velocity vector $\boldsymbol{\Omega}$.

First, as a hypothetical setup, we define the basis vectors of the laboratory frame (world coordinate system) (X, Y, Z) as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. The Galactic angular velocity is expressed as (see Equation 54)

$$\begin{aligned} \boldsymbol{\Omega} &\equiv \Omega^X\boldsymbol{i} + \Omega^Y\boldsymbol{j} + \Omega^Z\boldsymbol{k} = \Omega^I\boldsymbol{n}_I, \\ &\text{where } I \in (X, Y, Z), \\ &\boldsymbol{n}_I \in (\boldsymbol{n}_X, \boldsymbol{n}_Y, \boldsymbol{n}_Z) = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}). \end{aligned} \quad (33)$$

The transformation matrix of the basis vectors for an arbitrary direction of $\boldsymbol{\Omega}$ is given by Rodrigues' rotation formula,

$$\begin{aligned} K(\boldsymbol{\Omega}) &\equiv \boldsymbol{\delta} + \sin(\Omega t)n_\times(\boldsymbol{\Omega}) \\ &\quad +(1-\cos(\Omega t))[n_\times(\boldsymbol{\Omega})]^2 \end{aligned} \quad (34)$$

where, $\boldsymbol{\delta}$ is an unit tensor, and $n_\times$ is the matrix representation of outer product,

$$n_\times(\boldsymbol{\Omega}) \equiv \frac{1}{\Omega}\begin{pmatrix} 0 & -\Omega^Z & \Omega^Y \\ \Omega^Z & 0 & -\Omega^X \\ -\Omega^Y & \Omega^X & 0 \end{pmatrix}. \quad (35)$$

Its component-wise form is shown in Equation 36:

Thus, the transformation of the basis vectors of the non-inertial Cartesian system aligned with the Galaxy, i.e. the Galactic frame (x, y, z), is

$$\boldsymbol{n}_i = K(\boldsymbol{\Omega})\boldsymbol{n}_I \quad (37)$$

where $i \in (x, y, z)$ and $I \in (X, Y, Z)$. Here we assume that at $t = 0$, the laboratory frame (X, Y, Z) coincides with the Galactic frame (x, y, z).

From here on, we set the Galactic angular velocity vector to

$$\boldsymbol{\Omega} = \Omega\boldsymbol{n}_Z, \quad (38)$$

so that the Galactic-frame bases are

$$\begin{aligned} (\boldsymbol{n}_x \ \ \boldsymbol{n}_y \ \ \boldsymbol{n}_z) &= K(\boldsymbol{\Omega})(\boldsymbol{n}_X \ \ \boldsymbol{n}_Y \ \ \boldsymbol{n}_Z) \\ &= \begin{pmatrix} \cos(\Omega t) & -\sin(\Omega t) & 0 \\ \sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (39)$$

The inverse transformation is

$$\begin{aligned} (\boldsymbol{n}_X \ \ \boldsymbol{n}_Y \ \ \boldsymbol{n}_Z)_{\mathrm{G}} &= K(-\boldsymbol{\Omega})(\boldsymbol{n}_x \ \ \boldsymbol{n}_y \ \ \boldsymbol{n}_z) \\ &= \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (40)$$

For the basis vectors of the Galactic frame,

$$
\begin{aligned}
\boldsymbol{n}_x &= \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{G}}, \\
\boldsymbol{n}_y &= \begin{pmatrix} -\sin(\Omega t) \\ \cos(\Omega t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{\mathrm{G}} \\
\boldsymbol{n}_z &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\mathrm{G}},
\end{aligned}
\tag{41}
$$

where the subscript "G" in $\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}_{\mathrm{G}}$ denotes components in the Galactic frame.[4]

The time derivatives of these bases are

$$
\begin{aligned}
\dot{\boldsymbol{n}}_x &= \Omega \boldsymbol{n}_y, \\
\dot{\boldsymbol{n}}_y &= -\Omega \boldsymbol{n}_x, \\
\dot{\boldsymbol{n}}_z &= \boldsymbol{0}.
\end{aligned}
\tag{42}
$$

3D polar coordinates

We introduce the three-dimensional polar coordinate system (r, θ, φ) with respect to the galactic frame (x, y, z) as

$$
\boldsymbol{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{G}} = \begin{pmatrix} r\cos(\theta)\cos(\varphi) \\ r\cos(\theta)\sin(\varphi) \\ r\sin(\theta) \end{pmatrix}_{\mathrm{G}}.
\tag{43}
$$

The basis vectors are defined as

$$
\boldsymbol{e}_r \equiv \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial r}\right|}\frac{\partial \boldsymbol{r}}{\partial r}, \quad \boldsymbol{e}_\theta \equiv \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial \theta}\right|}\frac{\partial \boldsymbol{r}}{\partial \theta}, \quad \boldsymbol{e}_\varphi \equiv \frac{1}{\left|\frac{\partial \boldsymbol{r}}{\partial \varphi}\right|}\frac{\partial \boldsymbol{r}}{\partial \varphi}, \tag{44}
$$

namely,

$$
\begin{aligned}
\boldsymbol{e}_r &= \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ \sin(\theta) \end{pmatrix}_{\mathrm{G}} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}}, \\
\boldsymbol{e}_\theta &= \begin{pmatrix} -\sin(\theta)\cos(\varphi) \\ -\sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}_{\mathrm{G}} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{\mathrm{polar}}, \\
\boldsymbol{e}_\varphi &= \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}_{\mathrm{G}} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\mathrm{polar}}.
\end{aligned}
\tag{45}
$$

The cross products of these basis vectors are expressed as

$$
\begin{aligned}
\boldsymbol{e}_r \times \boldsymbol{e}_\theta &= -\boldsymbol{e}_\theta \times \boldsymbol{e}_r = -\boldsymbol{e}_\varphi \\
\boldsymbol{e}_\theta \times \boldsymbol{e}_\varphi &= -\boldsymbol{e}_\varphi \times \boldsymbol{e}_\theta = -\boldsymbol{e}_r \\
\boldsymbol{e}_\varphi \times \boldsymbol{e}_r &= -\boldsymbol{e}_r \times \boldsymbol{e}_\varphi = -\boldsymbol{e}_\theta.
\end{aligned}
\tag{46}
$$

Accordingly, the cross product of two arbitrary vectors can be written as

$$
\begin{aligned}
\boldsymbol{A} \times \boldsymbol{B} &= \left(A^r \boldsymbol{e}_r + A^\theta \boldsymbol{e}_\theta + A^\varphi \boldsymbol{e}_\varphi\right) \\
&\quad \times \left(B^r \boldsymbol{e}_r + B^\theta \boldsymbol{e}_\theta + B^\varphi \boldsymbol{e}_\varphi\right) \\
&= -\left(A^\theta B^\varphi - A^\varphi B^\theta\right)\boldsymbol{e}_r \\
&\quad -\left(A^\varphi B^r - A^r B^\varphi\right)\boldsymbol{e}_\theta \\
&\quad -\left(A^r B^\theta - A^\theta B^\varphi\right)\boldsymbol{e}_\varphi.
\end{aligned}
\tag{47}
$$

The time derivative of the basis vectors can be expressed, using the notation Equation 54, as

$$
\begin{gathered}
\dot{\boldsymbol{e}}_\alpha = \dot{\theta}\frac{\partial}{\partial \theta}\boldsymbol{e}_\alpha + \dot{\varphi}\frac{\partial}{\partial \varphi}\boldsymbol{e}_\alpha + \frac{\partial}{\partial t}\boldsymbol{e}_\alpha, \\
\text{where } \alpha \in (r, \theta, \varphi).
\end{gathered}
\tag{48}
$$

Here, $\frac{\partial}{\partial t}$ denotes the derivative with respect to explicit time dependence, acting on the galactic frame basis vectors $\boldsymbol{n}_i = \boldsymbol{n}_i(t), i \in (x, y, z)$. Since $\boldsymbol{e}_\alpha = (e_\alpha)^i \boldsymbol{n}_i(t)$, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\boldsymbol{e}_\alpha &= \frac{\partial}{\partial t}\left((e_\alpha)^i \boldsymbol{n}_i(t)\right) \\
&= (e_\alpha)^i \frac{\partial}{\partial t}\boldsymbol{n}_i(t)) \\
&= \Omega\left((e_\alpha)^x \boldsymbol{n}_y - (e_\alpha)^y \boldsymbol{n}_x\right) \\
&= \Omega \begin{pmatrix} -(e_\alpha)^y \\ (e_\alpha)^x \\ 0 \end{pmatrix}_{\mathrm{G}}.
\end{aligned}
\tag{49}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial t}\boldsymbol{e}_r &= \Omega \begin{pmatrix} -\cos(\theta)\sin(\varphi) \\ \cos(\theta)\cos(\varphi) \\ 0 \end{pmatrix}_{\mathrm{G}} = \Omega\cos(\theta)\boldsymbol{e}_\varphi, \\
\frac{\partial}{\partial t}\boldsymbol{e}_\theta &= \Omega \begin{pmatrix} \sin(\theta)\sin(\varphi) \\ -\sin(\theta)\cos(\varphi) \\ 0 \end{pmatrix}_{\mathrm{G}} = -\Omega\sin(\theta)\boldsymbol{e}_\varphi, \\
\frac{\partial}{\partial t}\boldsymbol{e}_\varphi &= \Omega \begin{pmatrix} -\cos(\varphi) \\ -\sin(\varphi) \\ 0 \end{pmatrix}_{\mathrm{G}} = -\Omega\left(\cos(\theta)\boldsymbol{e}_r - \sin(\theta)\boldsymbol{e}_\theta\right).
\end{aligned}
\tag{50}
$$

Thus, the time derivatives of the basis vectors are given by

[4] The subscript "G" indicates representation in the Galactic frame.

$$
\begin{aligned}
\dot{\boldsymbol{e}}_r &= \dot{\theta}\frac{\partial}{\partial\theta}\boldsymbol{e}_r + \dot{\varphi}\frac{\partial}{\partial\varphi}\boldsymbol{e}_r + \frac{\partial}{\partial t}\boldsymbol{e}_r \\
&= \dot{\theta}\boldsymbol{e}_\theta + (\dot{\varphi}+\Omega)\cos(\theta)\boldsymbol{e}_\varphi,
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_\theta &= \dot{\theta}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \dot{\varphi}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\theta + \frac{\partial}{\partial t}\boldsymbol{e}_\theta \\
&= -\dot{\theta}\boldsymbol{e}_r - (\dot{\varphi}+\Omega)\sin(\theta)\boldsymbol{e}_\varphi,
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_\varphi &= \dot{\varphi}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial}{\partial t}\boldsymbol{e}_\varphi \\
&= -(\dot{\varphi}+\Omega)(\cos(\theta)\boldsymbol{e}_r - \sin(\theta)\boldsymbol{e}_\theta).
\end{aligned} \tag{53}
$$

Covariant and contravariant

For an arbitrary vector $\boldsymbol{A}$, we express it as

$$\boldsymbol{A} = A^\alpha \boldsymbol{e}_\alpha, \tag{54}$$

where covariant components are denoted with lower indices and contravariant components with upper indices, following Einstein's summation convention over repeated indices.

For a three-dimensional coordinate vector $\boldsymbol{q}$, we impose a temporary rule for notation. In Cartesian coordinates, we write $\boldsymbol{q} = (x, y, z) = (x^1, x^2, x^3)$, that is, using basis vectors, $\boldsymbol{q} = x^i \boldsymbol{n}(i), i = 1, 2, 3$. On the other hand, in polar coordinates we write $\boldsymbol{q} = (r, \theta, \varphi) = (r^1, r^2, r^3)$, i.e., $\boldsymbol{q} = r^\alpha \boldsymbol{e}(\alpha), \alpha = 1, 2, 3$. By this definition, we obtain

$$\boldsymbol{q} = x^i \boldsymbol{n}_i = r^\alpha \boldsymbol{e}_\alpha. \tag{55}$$

The transformation between the basis vectors can then be written as

$$
\begin{aligned}
&\boldsymbol{e}_\alpha = R_\alpha{}^i \boldsymbol{n}_i, \quad \boldsymbol{n}_i = R_i^\alpha \boldsymbol{e}_\alpha \\
&\text{where } R_i^\alpha = \left(R_\alpha{}^i\right)^{-1}, \text{or } R_\alpha{}^i = \left(R_i{}^\alpha\right)^{-1}.
\end{aligned} \tag{56}
$$

Here, we define the metric tensor as

$$
\begin{aligned}
g_{ij} &\equiv \frac{\partial \boldsymbol{q}}{\partial x^i} \cdot \frac{\partial \boldsymbol{q}}{\partial x^j} = \frac{\partial r^\alpha}{\partial x^i} \cdot \frac{\partial r^\alpha}{\partial x^j}, \\
g_{\alpha\beta} &\equiv \frac{\partial \boldsymbol{q}}{\partial r^\alpha} \cdot \frac{\partial \boldsymbol{q}}{\partial r^\beta} = \frac{\partial x^i}{\partial r^\alpha} \cdot \frac{\partial x^i}{\partial r^\beta}.
\end{aligned} \tag{57}
$$

From Equation 55, differentiating both sides gives

$$
\begin{aligned}
\boldsymbol{e}_\alpha &\propto \frac{\partial x^i}{\partial r^\alpha} \boldsymbol{n}_i, \\
\boldsymbol{n}_i &\propto \frac{\partial r^\alpha}{\partial x^i} \boldsymbol{e}_\alpha,
\end{aligned} \tag{58}
$$

However, since the basis vectors are taken as unit vectors, normalization is required. For instance, for $r^2 = \theta, \boldsymbol{e}(2) = \boldsymbol{e}(\theta)$, we compute

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{q}}{\partial \theta} \bigg/ \left|\frac{\partial \boldsymbol{q}}{\partial \theta}\right|. \tag{59}$$

This corresponds to the definition in Equation 44 and is consistent with the result in Equation 45. Here, it is obtained as

$$
\begin{aligned}
\left|\frac{\partial \boldsymbol{r}}{\partial \theta}\right| &= \left|\boldsymbol{n}_j \frac{\partial x^j}{\partial \theta}\right| \\
&= \sqrt{\frac{\partial x^j}{\partial \theta}\frac{\partial x^j}{\partial \theta}} \\
&= \sqrt{g_{\theta\theta}}.
\end{aligned} \tag{60}
$$

From Equation 56, the transformation between basis vectors $\boldsymbol{n}(i)$ and $\boldsymbol{e}(\alpha)$ is given by

$$\boldsymbol{e}_\alpha = R_\alpha{}^i \boldsymbol{n}_i, \tag{61}$$

or equivalently,

$$(\boldsymbol{e}_1 \;\; \boldsymbol{e}_2 \;\; \boldsymbol{e}_3) = \begin{pmatrix} R_1{}^1 & R_2{}^1 & R_3{}^1 \\ R_1{}^2 & R_2{}^2 & R_3{}^2 \\ R_1{}^3 & R_2{}^3 & R_3{}^3 \end{pmatrix} (\boldsymbol{n}_1 \;\; \boldsymbol{n}_2 \;\; \boldsymbol{n}_3), \tag{62}$$

where

$$R_\alpha{}^i \equiv \frac{\partial x^i}{\partial r^\alpha} \bigg/ \sqrt{g_{\alpha\alpha}}. \tag{63}$$

That is,

$$R_\alpha{}^i = \begin{pmatrix} \cos(\theta)\cos(\varphi) & -\sin(\theta)\cos(\varphi) & -\sin(\varphi) \\ \cos(\theta)\sin(\varphi) & -\sin(\theta)\sin(\varphi) & \cos(\varphi) \\ \sin(\theta) & \cos(\theta) & 0 \end{pmatrix} \tag{64}$$

Furthermore, considering the inverse transformation,

$$\boldsymbol{n}_i = \left(R_\alpha{}^i\right)^{-1} \boldsymbol{e}_\alpha = \boldsymbol{e}_\alpha R^\alpha{}_i, \tag{65}$$

where the inverse is defined as

$$
\begin{aligned}
R^\alpha{}_i &\equiv \left(R_\alpha{}^i\right)^{-1} \\
&= \frac{\partial r^\alpha}{\partial x^i} \bigg/ \sqrt{g_{ii}},
\end{aligned} \tag{66}
$$

thus,

$$R^\alpha{}_i = \begin{pmatrix} \cos(\theta)\cos(\varphi) & \cos(\theta)\sin(\varphi) & \sin(\theta) \\ -\sin(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) & \cos(\theta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{pmatrix}, \tag{67}$$

with the identities

$$
\begin{aligned}
R^\alpha{}_i R_\alpha{}^j &= R_\alpha{}^j R^\alpha{}_i = \delta_i^{\,j}, \\
R^\alpha{}_i R_\beta{}^i &= R_\beta{}^i R^\alpha{}_i = \delta^\alpha{}_\beta,
\end{aligned} \tag{68}
$$

where $\sqrt{g_{ii} g_{\alpha\alpha}} = 1, \forall i \in (x, y, z)$ and $\alpha \in (r, \theta, \varphi)$.

For the coordinate transformation of an arbitrary vector $\boldsymbol{A}$,

$$\boldsymbol{A} = A^{\alpha}\boldsymbol{e}_{\alpha} = A^{\alpha}R_{\alpha}{}^{i}\boldsymbol{n}_{i} = A^{i}\boldsymbol{n}_{i}, \tag{69}$$

which gives

$$A^{i} = A^{\alpha}R_{\alpha}{}^{i} \tag{70}$$

Conversely,

$$\boldsymbol{A} = A^{i}\boldsymbol{n}_{i} = A^{i}\left(R_{\alpha}{}^{i}\right)^{-1}\boldsymbol{e}_{\alpha} = A^{\alpha}\boldsymbol{e}_{\alpha} \tag{71}$$

so that, we obtain

$$A^{\alpha} = R^{\alpha}{}_{i}A^{i}. \tag{72}$$

For clarity of exposition, we temporarily adopted the notation that indices i, α take the values $1, 2, 3$. However, it is equally valid to write directly $i \in (x, y, z)$ and $\alpha \in (r, \theta, \varphi)$.

Inner and outer product

Using this covariant and contravariant notation, inner and outer products of vectors can be expressed compactly.

For the inner product of two arbitrary vectors $\boldsymbol{A}, \boldsymbol{B}$,

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= A^{i}\boldsymbol{n}_{i}\cdot B^{j}\boldsymbol{n}_{j}\\ &= A^{i}B^{j}\boldsymbol{n}_{i}\cdot\boldsymbol{n}_{j}\\ &= A^{i}B^{j}\delta_{ij}\\ &= A^{i}B^{i}.\end{aligned} \tag{73}$$

Similarly, in polar coordinates, the basis vectors $\boldsymbol{e}(\alpha)$ are also orthonormal,

$$\boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}_{\beta} = \delta_{\alpha\beta}, \tag{74}$$

thus,

$$\boldsymbol{A}\cdot\boldsymbol{B} = A^{\alpha}B^{\alpha}. \tag{75}$$

For the outer product,

$$\boldsymbol{A}\times\boldsymbol{B} = A^{i}B^{j}\boldsymbol{n}_{i}\times\boldsymbol{n}_{j} = A^{i}B^{j}\varepsilon_{ijk}\boldsymbol{n}_{k}, \tag{76}$$

where the Levi-Civita symbol $\varepsilon(ijk)$ is introduced. In the chosen polar coordinate system, with the correspondence $(\alpha, \beta, \gamma) \in (r, \theta, \varphi) \to (1, 2, 3)$, we obtain

$$\boldsymbol{A}\times\boldsymbol{B} = -A^{\alpha}B^{\beta}\varepsilon_{\alpha\beta\gamma}\boldsymbol{e}_{\gamma}. \tag{77}$$

Angular velocity of milky way

Up to this point, the angular velocity of the Galaxy is written as

$$\begin{aligned}&\boldsymbol{\Omega} = \Omega^{I}\boldsymbol{n}_{I} = \Omega^{i}\boldsymbol{n}_{i} = \Omega^{\alpha}\boldsymbol{e}_{\alpha},\\ &\text{where, } I \in (X, Y, Z), i \in (x, y, z), \alpha \in (r, \theta, \varphi).\end{aligned} \tag{78}$$

Here, it is assumed to be the constant vector

$$\begin{aligned}\boldsymbol{\Omega} &= \begin{pmatrix}\Omega^{X}\\ \Omega^{Y}\\ \Omega^{Z}\end{pmatrix} = \begin{pmatrix}0\\0\\ \Omega\end{pmatrix}\\ &= \begin{pmatrix}\Omega^{x}\\ \Omega^{y}\\ \Omega^{z}\end{pmatrix}_{\mathrm{G}} = \begin{pmatrix}0\\0\\ \Omega\end{pmatrix}_{\mathrm{G}}.\end{aligned} \tag{79}$$

The transformation to curvilinear coordinates is given by

$$\Omega^{\alpha} = R^{\alpha}{}_{i}\Omega^{i} = \left(R_{\alpha}{}^{i}\right)^{-1}\Omega^{i}, \tag{80}$$

so that

$$\begin{aligned}&\boldsymbol{\Omega} =\\ &\begin{pmatrix}\cos(\theta)\cos(\varphi) & \cos(\theta)\sin(\varphi) & \sin(\theta)\\ -\sin(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) & \cos(\theta)\\ -\sin(\varphi) & \cos(\varphi) & 0\end{pmatrix}\begin{pmatrix}0\\0\\ \Omega\end{pmatrix}\\ &= \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\\ 0\end{pmatrix}_{\mathrm{polar}}\Omega.\end{aligned} \tag{81}$$

That is,

$$\Omega^{r} = \Omega\sin(\theta), \quad \Omega^{\theta} = \Omega\cos(\theta), \quad \Omega^{\varphi} = 0. \tag{82}$$

Coordinates of solar system

From Equation 43, let us express the position vector of the solar system in both Cartesian and polar coordinates of the galactic frame:

$$\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}_{\mathrm{G}} = \begin{pmatrix}r\\0\\0\end{pmatrix}_{\mathrm{polar}}. \tag{83}$$

We now examine the expressions for velocity and acceleration in each representation.

First, the velocity vector $\boldsymbol{v}$ in Cartesian coordinates of the galactic frame can be obtained using the time derivative of the basis vectors shown in Equation 42:

$$\begin{aligned}&\boldsymbol{v} \equiv \dot{\boldsymbol{r}}\\ &\quad= (\dot{x} - \Omega y)\boldsymbol{n}_{x} + (\dot{y} + \Omega x)\boldsymbol{n}_{y} + \dot{z}\boldsymbol{n}_{z}.\end{aligned} \tag{84}$$

That is,

$$\boldsymbol{v} = \begin{pmatrix}\dot{x} - \Omega y\\ \dot{y} + \Omega x\\ \dot{z}\end{pmatrix}_{\mathrm{G}}, \tag{85}$$

Since

$$\boldsymbol{\Omega} \times \boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}_{\mathrm{G}} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{G}} = \begin{pmatrix} -\Omega y \\ \Omega x \\ 0 \end{pmatrix}_{G}, \tag{86}$$

we have

$$\boldsymbol{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}_{\mathrm{G}} + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{87}$$

In polar coordinates of the galactic frame, the position vector $\boldsymbol{r}$, its time derivative $\boldsymbol{v}$, and the acceleration vector $\boldsymbol{a}$ are expressed as

$$\begin{aligned} \boldsymbol{r} &= r\boldsymbol{e}_r, \\ \boldsymbol{v} &\equiv \dot{\boldsymbol{r}} \\ &= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \\ &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r(\dot{\varphi} + \Omega)\cos(\theta)\boldsymbol{e}_\varphi, \end{aligned} \tag{88}$$

$$\begin{aligned} \boldsymbol{a} &= \ddot{\boldsymbol{r}} \\ &= \left(\ddot{r} - r\dot{\theta}^2 - r(\dot{\varphi} + \Omega)^2\cos^2(\theta)\right)\boldsymbol{e}_r \\ &\quad + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} + r(\dot{\varphi} + \Omega)^2\cos(\theta)\sin(\theta)\right)\boldsymbol{e}_\theta \\ &\quad + \left(r\ddot{\varphi}\cos(\theta) + 2(\dot{\varphi} + \Omega)\left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)\right)\boldsymbol{e}_\varphi. \end{aligned} \tag{89}$$

In the matrix form,

$$\boldsymbol{r} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}}, \tag{90}$$

$$\boldsymbol{v} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r(\dot{\varphi} + \Omega)\cos(\theta) \end{pmatrix}_{\mathrm{polar}}, \tag{91}$$

$$\boldsymbol{a} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r(\dot{\varphi} + \Omega)^2\cos^2(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + r(\dot{\varphi} + \Omega)^2\cos(\theta)\sin(\theta) \\ r\ddot{\varphi}\cos(\theta) + 2(\dot{\varphi} + \Omega)\left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right) \end{pmatrix}_{\mathrm{polar}} \tag{92}$$

where

$$\ddot{\boldsymbol{r}} = \ddot{r}\boldsymbol{e}_r + 2\dot{r}\dot{\boldsymbol{e}}_r + r\ddot{\boldsymbol{e}}_r, \tag{93}$$

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta + (\dot{\varphi} + \Omega)\cos(\theta)\boldsymbol{e}_\varphi, \tag{94}$$

$$\begin{aligned} \ddot{\boldsymbol{e}}_r &= \left[-\dot{\theta}^2 - (\dot{\varphi} + \Omega)^2\cos^2(\theta)\right]\boldsymbol{e}_r \\ &\quad + \left[\ddot{\theta} + (\dot{\varphi} + \Omega)^2\cos(\theta)\sin(\theta)\right]\boldsymbol{e}_\theta \\ &\quad + \left[-2\dot{\theta}(\dot{\varphi} + \Omega)\sin(\theta) + \ddot{\varphi}\cos(\theta)\right]\boldsymbol{e}_\varphi. \end{aligned} \tag{95}$$

Equation of motion. The equations of motion are expressed in terms of the generalized coordinates q by the Euler–Lagrange equation. Since we need to use both Cartesian and polar coordinate representations depending on the situation, we describe each formulation below.

Euler-Lagrange equation

For the Lagrangian

$$\begin{aligned} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &\equiv T(\boldsymbol{v}(\dot{\boldsymbol{q}}, \boldsymbol{q})) - m\Phi(\boldsymbol{q}), \\ T(\boldsymbol{v}(\dot{\boldsymbol{q}}, \boldsymbol{q})) &\equiv \frac{1}{2}m\boldsymbol{v}^2, \end{aligned} \tag{96}$$

where, the action

$$\mathcal{S} \equiv \int_{t_A}^{t_B} \mathcal{L}dt \tag{97}$$

is minimized:

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_A}^{t_B} \delta\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})dt \\ &\approx \int_{t_A}^{t_B} \left(\frac{\partial\mathcal{L}}{\partial q^i}\delta q^i + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\delta\dot{q}^i\right)dt \\ &= \int_{t_A}^{t_B} \left(\frac{\partial\mathcal{L}}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right)\delta q^i dt + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\delta q^i\bigg|_{t=t_A}^{t=t_B} \\ &= 0, \end{aligned} \tag{98}$$

which is the principle of least action. Hence the Euler–Lagrange equation follows:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \frac{\partial\mathcal{L}}{\partial q^i} \approx 0 \\ \Leftrightarrow\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}^i} - \frac{\partial T}{\partial q^i} - F^i \approx 0, \end{aligned} \tag{99}$$

where,

$$F^i \equiv -m\frac{\partial\Phi(\boldsymbol{q})}{\partial q^i} \tag{100}$$

denotes the gravitational force. The boundary term

$$\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\delta q^i\bigg|_{t=t_A}^{t=t_B} \approx 0, \tag{101}$$

implies the approximation that stellar radiation stresses or collisions with extrasolar objects are neglected.

Equation of motion for Cartesian coordinates

Let us express the equations of motion of the solar system in the galactic frame using Cartesian coordinates (x, y, z). From the velocity expression (see Equation 85),

$$v^2 = (\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2 + \dot{z}^2, \tag{102}$$

the kinetic energy is

$$
\begin{aligned}
\dot{x} &= \dot{r}\cos(\theta)\cos(\varphi) - r\dot{\theta}\sin(\theta)\cos(\varphi) - r\dot{\varphi}\cos(\theta)\sin(\varphi),\\
\dot{y} &= \dot{r}\cos(\theta)\sin(\varphi) - r\dot{\theta}\sin(\theta)\sin(\varphi) + r\dot{\varphi}\cos(\theta)\cos(\varphi),\\
\dot{z} &= \dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\ddot{x} &= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\right)\cos(\theta)\cos(\varphi) + \left(-2\dot{r}\dot{\theta} - r\ddot{\theta}\right)\sin(\theta)\cos(\varphi)\\
&\quad +(-2\dot{r}\dot{\varphi} - r\ddot{\varphi})\cos(\theta)\sin(\varphi) + 2r\dot{\theta}\dot{\varphi}\sin(\theta)\sin(\varphi),\\
\ddot{y} &= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\right)\cos(\theta)\sin(\varphi) + \left(-2\dot{r}\dot{\theta} - r\ddot{\theta}\right)\sin(\theta)\sin(\varphi)\\
&\quad +(2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos(\theta)\cos(\varphi) - 2r\dot{\theta}\dot{\varphi}\sin(\theta)\cos(\varphi),\\
\ddot{z} &= \left(\ddot{r} - r\dot{\theta}^2\right)\sin(\theta) + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos(\theta)
\end{aligned} \tag{109}
$$

$$
T = \frac{1}{2}m\left[(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2 + \dot{z}^2\right]. \tag{103}
$$

Differentiating yields

$$
\begin{aligned}
\frac{\partial}{\partial x}T &= m\Omega(\dot{y} + \Omega x),\\
\frac{\partial}{\partial y}T &= -m\Omega(\dot{x} - \Omega y),\\
\frac{\partial}{\partial z}T &= 0,
\end{aligned} \tag{104}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}T &= m(\ddot{x} - \Omega\dot{y}),\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{y}}T &= m(\ddot{y} + \Omega\dot{x}),\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{z}}T &= m\ddot{z}.
\end{aligned} \tag{105}
$$

Thus, by the Euler–Lagrange equation,

$$
m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix}_{\mathrm{G}} = 2m\Omega\begin{pmatrix}\dot{y}\\ -\dot{x}\\ 0\end{pmatrix}_{\mathrm{G}} + m\Omega^2\begin{pmatrix}x\\ y\\ 0\end{pmatrix}_{\mathrm{G}} + \boldsymbol{F} \tag{106}
$$

where the right-hand side represents the Coriolis force (first term), the centrifugal force (second term), and gravity (third term).

Equation of motion for polar coordinates

Next, we describe the equations of motion in the galactic frame using polar coordinates (r, θ, φ). Using the coordinate relations for $\boldsymbol{r}$, the velocity and acceleration components are obtained, which upon substitution into the Cartesian equations yield the polar form:

$$
\boldsymbol{r} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}_{\mathrm{G}} = r\boldsymbol{e}_r = \begin{pmatrix}r\cos(\theta)\cos(\varphi)\\ r\cos(\theta)\sin(\varphi)\\ r\sin(\theta)\end{pmatrix}_{\mathrm{G}}. \tag{107}
$$

On the other hand, the same equations of motion can be derived directly from the Lagrangian in polar coordinates, as well as Equation 108 and Equation 109:

By substituting these expressions into Equation 106 and rewriting in polar coordinates, while paying careful attention to the component representations of each basis vector in Equation 45, we obtain

$$
\begin{aligned}
\begin{pmatrix}x\\ y\\ 0\end{pmatrix}_{\mathrm{G}} &= r\cos(\theta)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}_{\mathrm{G}}\\
&= r\left(\cos^2(\theta)\boldsymbol{e}_r - \cos(\theta)\sin(\theta)\boldsymbol{e}_\theta\right)\\
&= r\cos(\theta)\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\\ 0\end{pmatrix}_{\mathrm{polar}},
\end{aligned} \tag{110}
$$

$$
\begin{aligned}
\begin{pmatrix}\dot{y}\\ -\dot{x}\\ 0\end{pmatrix}_{\mathrm{G}} &= \left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)\begin{pmatrix}\sin(\varphi)\\ -\cos(\varphi)\\ 0\end{pmatrix}_{\mathrm{G}}\\
&\quad + r\dot{\varphi}\cos(\theta)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}_{\mathrm{G}}\\
&= r\dot{\varphi}\cos^2(\theta)\boldsymbol{e}_r - r\dot{\varphi}\cos(\theta)\sin(\theta)\boldsymbol{e}_\theta\\
&\quad - \left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)\boldsymbol{e}_\varphi\\
&= \begin{pmatrix}r\dot{\varphi}\cos^2(\theta)\\ -r\dot{\varphi}\cos(\theta)\sin(\theta)\\ -\dot{r}\cos(\theta) + r\dot{\theta}\sin(\theta)\end{pmatrix}_{\mathrm{polar}},
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} &= \left(\ddot{r}-r\dot{\theta}^2\right)\begin{pmatrix}\cos(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi)\\ \sin(\theta)\end{pmatrix}_{\mathrm{G}} \\
&\quad+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\begin{pmatrix}-\sin(\theta)\cos(\varphi)\\ -\sin(\theta)\sin(\varphi)\\ \cos(\theta)\end{pmatrix}_{\mathrm{G}} \\
&\quad-r\dot{\varphi}^2\cos(\theta)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}_{\mathrm{G}} \\
&\quad+\left[(2\dot{r}\dot{\varphi}+r\ddot{\varphi})\cos(\theta)-2r\dot{\theta}\dot{\varphi}\sin(\theta)\right]\begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\\ 0\end{pmatrix}_{\mathrm{G}} \\
&= \left(\ddot{r}-r\dot{\theta}^2\right)\boldsymbol{e}_r+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\boldsymbol{e}_\theta \\
&\quad-r\dot{\varphi}^2(\cos^2(\theta)\boldsymbol{e}_r-\cos(\theta)\sin(\theta)\boldsymbol{e}_\theta) \\
&\quad+\left[(2\dot{r}\dot{\varphi}+r\ddot{\varphi})\cos(\theta)-2r\dot{\theta}\dot{\varphi}\sin(\theta)\right]\boldsymbol{e}_\varphi \\
&= \begin{pmatrix}\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\cos^2(\theta)\\ 2\dot{r}\dot{\theta}+r\ddot{\theta}+r\dot{\varphi}^2\cos(\theta)\sin(\theta)\\ (2\dot{r}\dot{\varphi}+r\ddot{\varphi})\cos(\theta)-2r\dot{\theta}\dot{\varphi}\sin(\theta)\end{pmatrix}_{\text{polar}},
\end{aligned}
\tag{112}
$$

so that,

$$
\begin{aligned}
&m\begin{pmatrix}\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\cos^2(\theta)\\ 2\dot{r}\dot{\theta}+r\ddot{\theta}+r\dot{\varphi}^2\cos(\theta)\sin(\theta)\\ (2\dot{r}\dot{\varphi}+r\ddot{\varphi})\cos(\theta)-2r\dot{\theta}\dot{\varphi}\sin(\theta)\end{pmatrix}_{\text{polar}} \\
&\quad= 2m\Omega\begin{pmatrix}r\dot{\varphi}\cos^2(\theta)\\ -r\dot{\varphi}\cos(\theta)\sin(\theta)\\ -\dot{r}\cos(\theta)+r\dot{\theta}\sin(\theta)\end{pmatrix}_{\text{polar}} \\
&\qquad+m\Omega^2 r\cos(\theta)\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\\ 0\end{pmatrix}_{\text{polar}}+\boldsymbol{F}.
\end{aligned}
\tag{113}
$$

On the other hand, the same equations of motion can be obtained from the Lagrangian. From Equation 91,

$$
v^2=\dot{r}^2+\left(r\dot{\theta}\right)^2+r^2(\dot{\varphi}+\Omega)^2\cos^2(\theta),
\tag{114}
$$

so that the kinetic energy in polar coordinates is

$$
T=\frac{1}{2}m\left(\dot{r}^2+\left(r\dot{\theta}\right)^2+r^2(\dot{\varphi}+\Omega)^2\cos^2(\theta)\right),
\tag{115}
$$

which gives

$$
\begin{aligned}
\frac{\partial}{\partial r}T &= mr\left(\dot{\theta}^2+(\dot{\varphi}+\Omega)^2\cos^2(\theta)\right), \\
\frac{\partial}{\partial \theta}T &= -mr^2(\dot{\varphi}+\Omega)^2\sin(\theta)\cos(\theta), \\
\frac{\partial}{\partial \varphi}T &= 0,
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{r}}T &= m\ddot{r}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\theta}}T &= mr\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right), \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\varphi}}T &= \frac{\mathrm{d}}{\mathrm{d}t}(mr^2(\dot{\varphi}+\Omega)\cos^2(\theta)) \\
&= mr\cos(\theta)\left(2\dot{r}(\dot{\varphi}+\Omega)\cos(\theta)+r\ddot{\varphi}\cos(\theta)\right. \\
&\qquad\left.-2r\dot{\theta}(\dot{\varphi}+\Omega)\sin(\theta)\right).
\end{aligned}
\tag{117}
$$

From the Euler–Lagrange equations, one finds

$$
\begin{aligned}
m\ddot{r} &= mr\left(\dot{\theta}^2+(\dot{\varphi}+\Omega)^2\cos^2(\theta)\right) \\
&\quad-m\frac{\partial}{\partial r}\Phi(r,\theta,\varphi),
\end{aligned}
\tag{118}
$$

$$
\begin{aligned}
mr\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) &= -mr^2(\dot{\varphi}+\Omega)^2\sin(\theta)\cos(\theta) \\
&\quad-m\frac{\partial}{\partial \theta}\Phi(r,\theta,\varphi),
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
&mr\cos(\theta)\left(2\dot{r}(\dot{\varphi}+\Omega)\cos(\theta)+r\ddot{\varphi}\cos(\theta)\right. \\
&\quad\left.-2r\dot{\theta}(\dot{\varphi}+\Omega)\sin(\theta)\right) \\
&= -m\frac{\partial}{\partial \varphi}\Phi(r,\theta,\varphi)
\end{aligned}
\tag{120}
$$

which are equivalent to the respective directional components of Equation 113.

Multipole expansion. For the integral in the potential Equation 2,

$$
\int_V \frac{\rho}{|\boldsymbol{r}-\boldsymbol{x}|}d\boldsymbol{x}
\tag{121}
$$

using the notation introduced in Equation 54, we write

$$
\boldsymbol{\xi}=\xi^i\boldsymbol{n}_i, \xi=|\boldsymbol{\xi}|=\sqrt{\xi^i\xi^i},
\tag{122}
$$

and define

$$
\begin{aligned}
f(\boldsymbol{r},\boldsymbol{x}) &\equiv \frac{1}{|\boldsymbol{r}-\boldsymbol{x}|}=\frac{1}{r}g(\boldsymbol{\varepsilon}), \\
g(\boldsymbol{\varepsilon}) &\equiv \frac{1}{\xi}=\frac{1}{|\hat{\boldsymbol{r}}-\boldsymbol{\varepsilon}|}.
\end{aligned}
\tag{123}
$$

Here,

$$
\begin{aligned}
\xi^i &= \hat{r}^i - \varepsilon^i, \\
|\boldsymbol{\xi}|^2 &= \xi^i \xi^i \\
&= \hat{r}^i \hat{r}^i - 2\hat{r}^i \varepsilon^i + \varepsilon^i \varepsilon^i \\
&= 1 - 2\hat{r}^i \varepsilon^i + \varepsilon^i \varepsilon^i
\end{aligned} \tag{124}
$$

Since $x \ll r$, we set

$$
\varepsilon^i \equiv \frac{x^i}{r} \ll 1 \tag{125}
$$

and expand the function $g(\boldsymbol{\varepsilon})$:

$$
\begin{aligned}
g(\boldsymbol{\varepsilon}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\varepsilon^i \frac{\partial}{\partial \varepsilon^i} \right)^n g(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\mathbf{0}} \\
&= g(0) + \varepsilon^i \frac{\partial}{\partial \varepsilon^i} g(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\mathbf{0}} \\
&\quad + \frac{1}{2!} \varepsilon^i \varepsilon^j \frac{\partial^2}{\partial \varepsilon^i \partial \varepsilon^j} g(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\mathbf{0}} + \cdots,
\end{aligned} \tag{126}
$$

and,

$$
\begin{aligned}
g(\boldsymbol{\varepsilon}) &\equiv \frac{1}{\xi} = \frac{1}{\sqrt{1 - 2\hat{r}^i \varepsilon^i + \varepsilon^i \varepsilon^i}}, \\
\frac{\partial}{\partial \varepsilon^j} g(\boldsymbol{\varepsilon}) &= \frac{\xi^j}{\xi^3}, \\
\frac{\partial^2}{\partial \varepsilon^j \partial \varepsilon^k} g(\boldsymbol{\varepsilon}) &= -\frac{\delta^j_k}{\xi^3} + 3 \frac{\xi^j \xi^k}{\xi^5}, \\
&\vdots
\end{aligned} \tag{127}
$$

This yields

$$
\begin{aligned}
g(\mathbf{0}) &= 1, \\
\frac{\partial}{\partial \varepsilon^i} g(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\mathbf{0}} &= \hat{r}^i, \\
\frac{\partial^2}{\partial \varepsilon^i \partial \varepsilon^j} g(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\mathbf{0}} &= -\delta^j_k + 3\hat{r}^j \hat{r}^k, \\
&\vdots
\end{aligned} \tag{128}
$$

Thus,

$$
g(\boldsymbol{\varepsilon}) = 1 + \hat{r}^i \varepsilon^i + \frac{1}{2} \left(3\hat{r}^j \hat{r}^k - \delta^j_k \right) \varepsilon^j \varepsilon^k + \cdots. \tag{129}
$$

Substituting back gives

$$
\begin{aligned}
f(\boldsymbol{r}, \boldsymbol{x}) &= \frac{1}{r} + \frac{r^i x^i}{r^3} + \frac{3r^j r^k x^j x^k - r^2 x^2}{2r^5} + \cdots \\
&= \frac{1}{r} + \frac{r^i}{r^3} x^i + \frac{r^j r^k}{2r^5} \left(3x^j x^k - \delta^j_k x^2 \right) + \cdots,
\end{aligned} \tag{130}
$$

Hence, the multipole expansion is

$$
\int_V \frac{\rho}{|\boldsymbol{r} - \boldsymbol{x}|} d\boldsymbol{x} = \frac{M}{r} + \frac{r^i}{r^3} P^i + \frac{r^j r^k}{2r^5} Q^{jk} + \cdots \tag{131}
$$

where

$$
\begin{aligned}
M &= \int_V \rho d\boldsymbol{x}, \\
P^i &= \int_V \rho x^i d\boldsymbol{x}, \\
Q^{jk} &= \int_V \rho \left(3x^j x^k - \delta^j_k x^2 \right) d\boldsymbol{x}, \\
&\vdots
\end{aligned} \tag{132}
$$

Gravitational potential

From the multipole expansion above shown in Equation 131, the gravitational potential is

$$
\begin{aligned}
\Phi(\boldsymbol{r}) &= \Phi^{(m)}(\boldsymbol{r}) + \Phi^{(q)}(\boldsymbol{r}) + \cdots \\
&= -G \left(\frac{M}{r} + \frac{1}{2} Q^{ij} \frac{r^i r^j}{r^5} + \cdots \right).
\end{aligned} \tag{133}
$$

Multipole expansions for the triaxial ellipsoid

Consider the triaxial ellipsoid,

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1. \tag{134}
$$

By symmetry of the homogeneous ellipsoid, odd-order multipole moments vanish. Monopole moment:

$$
M = \frac{4\pi}{3} \rho abc, \tag{135}
$$

Quadrupole moments:

$$
\begin{aligned}
Q^{xx} &= \frac{M}{5} \left(2a^2 - b^2 - c^2 \right) = \frac{M}{5} q^{xx}, \\
Q^{yy} &= \frac{M}{5} \left(2b^2 - a^2 - c^2 \right) = \frac{M}{5} q^{yy}, \\
Q^{zz} &= \frac{M}{5} \left(2c^2 - a^2 - b^2 \right) = \frac{M}{5} q^{zz}
\end{aligned} \tag{136}
$$

with all off-diagonal terms equal to zero.

Potential force

From the gravitational potential, the force due to each moment is

$$
\begin{aligned}
\boldsymbol{F} &= -m \boldsymbol{\nabla} \Phi(\boldsymbol{r}) \\
&= -m \boldsymbol{\nabla} \Phi^{(m)}(\boldsymbol{r}) - m \boldsymbol{\nabla} \Phi^{(q)}(\boldsymbol{r}) - \cdots \\
&= \boldsymbol{F}^{(m)} + \boldsymbol{F}^{(q)} + \cdots
\end{aligned} \tag{137}
$$

with

$$
\begin{aligned}
\boldsymbol{F} = F^i \boldsymbol{n}_i &= -\boldsymbol{n}_i m \frac{\partial}{\partial r^i} \Phi(\boldsymbol{r}), \\
i.e. \quad F^i &= -m \frac{\partial}{\partial r^i} \Phi(\boldsymbol{r})
\end{aligned} \tag{138}
$$

(see Equation 54). Here, monopole and quadrupole forces are

$$\begin{aligned}F^{(m),i} &= -m\frac{\partial}{\partial r^i}\Phi^{(m)}(\boldsymbol{r}) \\ &= -Gm\frac{M}{r^2}\hat{r}^i, \\ F^{(q),i} &= -\frac{\partial}{\partial r^i}\Phi^{(q)}(\boldsymbol{r}) \\ &= -Gm\frac{Q^{jk}}{2r^4}\left(\delta^j_i\hat{r}^k + \delta^k_i\hat{r}^j - 5\hat{r}^i\hat{r}^j\hat{r}^k\right) \\ &\vdots\end{aligned} \tag{139}$$

This expression is written in Cartesian coordinates (x, y, z), where

$$\hat{\boldsymbol{r}} = \boldsymbol{e}_r = \hat{r}^i\boldsymbol{n}_i = \begin{pmatrix}\cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ \sin(\theta)\end{pmatrix}_{\mathrm{G}}. \tag{140}$$

This transformation can be written as

$$\hat{r}^\alpha = R^\alpha{}_i\hat{r}^i, \tag{141}$$

where

$$R^\alpha{}_i = \begin{pmatrix}(\boldsymbol{e}_r)^t \\ (\boldsymbol{e}_\theta)^t \\ \left(\boldsymbol{e}_\varphi\right)^t\end{pmatrix}. \tag{142}$$

Thus,

$$\hat{\boldsymbol{r}} = \begin{pmatrix}(\boldsymbol{e}_r)^t \\ (\boldsymbol{e}_\theta)^t \\ \left(\boldsymbol{e}_\varphi\right)^t\end{pmatrix}\boldsymbol{e}_r = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}_{\mathrm{polar}} \tag{143}$$

can also be expressed in this form.

To make the orbit of the solar system around the galaxy easier to handle, we perform a transformation to polar coordinates (r, θ, φ). Noting that

$$\boldsymbol{F} = F^i\boldsymbol{n}_i = F^\alpha\boldsymbol{e}_\alpha \tag{144}$$

and using the transformation given in Equation 72, we obtain

$$F^\alpha = R^\alpha{}_iF^i = \left(R_\alpha{}^i\right)^{-1}F^i \tag{145}$$

which gives the contravariant components F^α in polar coordinates.

From this, let us calculate the forces due to each moment. For the monopole,

$$\begin{aligned}F^{(m),\alpha} &= R^\alpha{}_iF^{(m),i} \\ &= R^\alpha{}_i\left(-Gm\frac{M}{r^2}\hat{r}^i\right) \\ &= -Gm\frac{M}{r^2}\hat{r}^\alpha\end{aligned} \tag{146}$$

or equivalently,

$$\boldsymbol{F}^{(m)} = -Gm\frac{M}{r^2}\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}_{\mathrm{polar}} \tag{147}$$

For the quadrupole moment, let

$$\boldsymbol{F}^{(q)} = m\frac{G}{2r^4}\boldsymbol{f} \tag{148}$$

then

$$\begin{aligned}f^\alpha &= R^\alpha{}_if^i \\ &= R^\alpha{}_iQ^{jk}\left(\delta^j_i\hat{r}^k + \delta^k_i\hat{r}^j - 5\hat{r}^i\hat{r}^j\hat{r}^k\right) \\ &= R^\alpha{}_jQ^{jk}\hat{r}^k + R^\alpha{}_kQ^{jk}\hat{r}^j - 5\hat{r}^\alpha Q^{jk}\hat{r}^j\hat{r}^k.\end{aligned} \tag{149}$$

The components of this force can be computed as

$$\begin{aligned}&f^x \\ &= 2Q^{xx}\hat{r}^x \\ &\quad -5\left(Q^{xx}(\hat{r}^x)^2 + Q^{yy}(\hat{r}^y)^2 + Q^{zz}(\hat{r}^z)^2\right)\hat{r}^x \\ &= 2Q^{xx}\cos(\theta)\cos(\varphi) \\ &\quad -5(Q^{xx}\cos^2(\theta)\cos^2(\varphi) \\ &\qquad +Q^{yy}\cos^2(\theta)\sin^2(\varphi) \\ &\qquad +Q^{zz}\sin^2(\theta))\end{aligned} \tag{150}$$

and similarly

$$\begin{aligned}f^x = {}&\frac{2}{5}Mq^{xx}\cos(\theta)\cos(\varphi) \\ &-M(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\ &\quad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\ &\quad +q^{zz}\sin^2(\theta)),\end{aligned} \tag{151}$$

$$\begin{aligned}f^y = {}&\frac{2}{5}Mq^{yy}\cos(\theta)\sin(\varphi) \\ &-M(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\ &\quad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\ &\quad +q^{zz}\sin^2(\theta)),\end{aligned} \tag{152}$$

$$
\begin{aligned}
f^z = \frac{2}{5} & M q^{zz} \sin(\theta) \\
& -M\big(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\
& \quad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\
& \quad +q^{zz}\sin^2(\theta)\big).
\end{aligned}
\tag{153}
$$

In polar coordinates,

$$
\begin{aligned}
& f^r \\
& = 2\big({R^r}_x Q^{xx}\hat{r}^x + {R^r}_y Q^{yy}\hat{r}^y + {R^r}_z Q^{zz}\hat{r}^z\big) \\
& \quad -5\hat{r}^r\left(Q^{xx}(\hat{r}^x)^2 + Q^{yy}(\hat{r}^y)^2 + Q^{zz}(\hat{r}^z)^2\right) \\
& = -3\big(Q^{xx}\cos^2(\theta)\cos^2(\varphi) \\
& \qquad +Q^{yy}\cos^2(\theta)\sin^2(\varphi) \\
& \qquad +Q^{zz}\sin^2(\theta)\big)
\end{aligned}
\tag{154}
$$

and computing all components similarly, we obtain

$$
\begin{aligned}
f^r &= -\frac{3}{5}M\big(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\
&\quad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\
&\quad +q^{zz}\sin^2(\theta)\big) \\
f^\theta &= \frac{1}{5}M\sin(2\theta)\big(-q^{xx}\cos^2(\varphi) \\
&\quad -q^{yy}\sin^2(\varphi) + q^{zz}\big) \\
f^\varphi &= \frac{2}{5}M\big(-q^{xx}\cos(\theta)\cos(\varphi)\sin(\varphi) \\
&\quad +q^{yy}\sin(\theta)\cos(\varphi)\big).
\end{aligned}
\tag{155}
$$

Libration analysis. Here, we employ linear stability analysis to investigate the libration frequencies.

Let us first examine the trajectories that satisfy the equation of motion given in Equation 113. We begin by considering circular orbits in a plane, followed by the motion in the radial r direction and in the elevation θ direction, and finally the three-dimensional helical orbits combining these motions.

Circular motion

We first analyze the circular orbit on the (x,y)-plane:

$$
\begin{aligned}
&\dot{r} = 0, \text{ i.e}, r = r_0 \text{ is const.} \\
&\theta = \dot{\theta} = 0.
\end{aligned}
\tag{156}
$$

From Equation 113, we obtain

$$
\begin{aligned}
& m\begin{pmatrix} -r_0\dot{\varphi}^2 \\ 0 \\ r_0\ddot{\varphi} \end{pmatrix}_{\text{polar}} \\
& = 2m\Omega\begin{pmatrix} r_0\dot{\varphi} \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} + m r_0\Omega^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} + \boldsymbol{F}|_{\theta=\dot{\theta}=0}.
\end{aligned}
\tag{157}
$$

Furthermore, we consider only the gravitational force due to the monopole term:

$$
\boldsymbol{F} \approx \boldsymbol{F}^{(m)}.
\tag{158}
$$

Hence,

$$
\begin{aligned}
\begin{pmatrix} -r_0\dot{\varphi}^2 \\ 0 \\ r_0\ddot{\varphi} \end{pmatrix}_{\text{polar}} & \approx r_0\Omega\begin{pmatrix} 2\dot{\varphi}+\Omega \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} \\
& \qquad -G\frac{1}{r_0^2}\begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}_{\text{polar}},
\end{aligned}
\tag{159}
$$

so that,

$$
\ddot{\varphi} = 0, \text{i.e.,} \quad \dot{\varphi} = \omega_0 \text{ is const.}
\tag{160}
$$

and,

$$
\begin{aligned}
-r_0\omega_0^2 &= r_0(2\Omega\omega_0 + \Omega^2) - \frac{GM}{r_0^2}, \\
(\omega_0+\Omega)^2 r_0^3 &= GM, \\
r_0 &= \left(\frac{GM}{(\omega_0+\Omega)^2}\right)^{\frac{1}{3}} \text{ if } \omega_0 \neq -\Omega \\
&.
\end{aligned}
\tag{161}
$$

Therefore,

$$
r_0(\omega_0) = \frac{1}{\kappa(\omega_0)},
\tag{162}
$$

where,

$$
(\kappa(\omega))^3 \equiv \frac{(\omega_0+\Omega)^2}{GM}.
\tag{163}
$$

The circular orbit is then given by

$$
\begin{aligned}
\boldsymbol{r} = \boldsymbol{r}_0 &= \begin{pmatrix} r_0\cos(\omega_0 t + \varphi_0) \\ r_0\sin(\omega_0 t + \varphi_0) \\ 0 \end{pmatrix}_{\text{G}}, \\
& i.e., x_0^2 + y_0^2 = r_0^2,
\end{aligned}
\tag{164}
$$

where, φ_0 is const. Note that the special case $\omega_0 = -\Omega$ corresponds to a situation in which the entire solar system is at rest.

Elliptical orbit (Kepler's Problem)

Next, we turn to the more general solution, the elliptical orbit, which is also known as Kepler's problem [7].

- Equation of motion for elliptical orbit

As in the case of circular orbits, we examine the orbit on the (x,y)-plane with $\theta = \dot{\theta} = 0$:

$$\boldsymbol{r} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ 0 \end{pmatrix}_{\mathrm{G}} = r\boldsymbol{e}_r(\varphi), \tag{165}$$

where we assume $r = r(\varphi)$. Noting that the time derivatives of the basis vectors are

$$\dot{\boldsymbol{e}_r} = \dot{\varphi}\boldsymbol{e}_\varphi, \quad \dot{\boldsymbol{e}_\varphi} = -\dot{\varphi}\boldsymbol{e}_r \tag{166}$$

we find that, for the position vector $\boldsymbol{r}$,

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi. \tag{167}$$

$$\begin{aligned} \ddot{\boldsymbol{r}} &= (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\boldsymbol{e}_\varphi \\ &= (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi})\boldsymbol{e}_\varphi, \end{aligned} \tag{168}$$

thus,

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{r} \\ 0 \\ r\dot{\varphi} \end{pmatrix}_{\mathrm{polar}}, \quad \ddot{\boldsymbol{r}} = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 0 \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) \end{pmatrix}_{\mathrm{polar}}. \tag{169}$$

Here again, considering the equation of motion Equation 113 with gravity restricted to the monopole term, we obtain

$$\begin{aligned} &m\begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 0 \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) \end{pmatrix}_{\mathrm{polar}} \\ &= 2m\Omega\begin{pmatrix} r\dot{\varphi} \\ 0 \\ -\dot{r} \end{pmatrix}_{\mathrm{polar}} + m\Omega^2 r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}} \\ &\quad -G\frac{m}{r^2}\begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}}. \end{aligned} \tag{170}$$

- Angular momentum conservation

From the φ component of the equation of motion Equation 170, we obtain

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0, \quad \text{where } h \equiv r^2(\dot{\varphi} + \Omega), \tag{171}$$

which defines the conserved quantity h, corresponding to angular momentum. The angular momentum vector is defined as

$$\begin{aligned} \boldsymbol{h} &\equiv \boldsymbol{r} \times \boldsymbol{v} \\ &= \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}} \times \begin{pmatrix} \dot{r} \\ 0 \\ r(\dot{\varphi} + \Omega) \end{pmatrix}_{\mathrm{polar}} \\ &= \begin{pmatrix} 0 \\ r^2(\dot{\varphi} + \Omega) \\ 0 \end{pmatrix}_{\mathrm{polar}} \\ &= \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}_{\mathrm{polar}} \end{aligned} \tag{172}$$

where the velocity vector $\boldsymbol{v}$ is

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r} \\ &= \begin{pmatrix} \dot{r} \\ 0 \\ r\dot{\varphi} \end{pmatrix}_{\mathrm{polar}} + \begin{pmatrix} 0 \\ \Omega \\ 0 \end{pmatrix}_{\mathrm{polar}} \times \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}_{\mathrm{polar}} \\ &= \begin{pmatrix} \dot{r} \\ 0 \\ r(\dot{\varphi} + \Omega) \end{pmatrix}_{\mathrm{polar}}. \end{aligned} \tag{173}$$

This angular momentum vector is also conserved:

$$\dot{\boldsymbol{h}} = \frac{\mathrm{d}h}{\mathrm{d}t}\boldsymbol{e}_\theta + h\dot{\boldsymbol{e}}_\theta = \boldsymbol{0}. \tag{174}$$

- Energy conservation

On the other hand, concerning the r-direction of the equation of motion Equation 170, we obtain

$$\begin{aligned} \ddot{r} - r\dot{\varphi}^2 &= 2\Omega r\dot{\varphi} + r\Omega^2 - G\frac{M}{r^2} \\ r^2\ddot{r} &= r^3(\dot{\varphi} + \Omega)^2 - GM \\ r^2\ddot{r} &= \frac{h^2}{r} - GM. \end{aligned} \tag{175}$$

Here, the total energy E, i.e., the sum of kinetic energy T and potential Φ, is defined as

$$\begin{aligned} E &\equiv T + m\Phi \\ &= \frac{1}{2}mv^2 - Gm\frac{M}{r}, \end{aligned} \tag{176}$$

where

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2(\dot{\varphi} + \Omega)^2 \\ &= \dot{r}^2 + \frac{h^2}{r^2}, \end{aligned} \tag{177}$$

so that

$$E = \frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - Gm\frac{M}{r} \tag{178}$$

is obtained. Its time derivative is

$$\begin{aligned} \dot{E} &= m\dot{r}\ddot{r} - m\frac{h^2}{r^3}\dot{r} + Gm\frac{M}{r^2}\dot{r} \\ &= m\frac{\dot{r}}{r^2}\left(r^2\ddot{r} - \frac{h^2}{r} + GM\right) \end{aligned} \tag{179}$$

and from Equation 175,

$$\dot{E} = 0, \tag{180}$$

which shows that E is conserved.

- Laplace-Runge-Lenz vector

Let us define the eccentricity vector[5]

$$\boldsymbol{A} \equiv \boldsymbol{v} \times \boldsymbol{h} - GM\hat{\boldsymbol{r}} \tag{181}$$

As shown in Equation 172, $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$, and

$$\boldsymbol{v} \times \boldsymbol{r} \times \boldsymbol{v} = (v^2\boldsymbol{\delta} - \boldsymbol{v}\boldsymbol{v}) \cdot \boldsymbol{r}, \tag{182}$$

so that,

$$\boldsymbol{A} = \left[r(v^2\boldsymbol{\delta} - \boldsymbol{v}\boldsymbol{v}) - GM\boldsymbol{\delta}\right] \cdot \boldsymbol{e}_r \tag{183}$$

or equivalently,

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} \frac{h^2}{r} - GM & 0 & -h\dot{r} \\ 0 & 0 & 0 \\ -h\dot{r} & 0 & r\dot{r}^2 - GM \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} \\ &= \begin{pmatrix} \frac{h^2}{r} - GM \\ 0 \\ -h\dot{r} \end{pmatrix}_{\text{polar}} \end{aligned} \tag{184}$$

where,

$$\boldsymbol{v} = \begin{pmatrix} \dot{r} \\ 0 \\ \frac{h}{r} \end{pmatrix}_{\text{polar}} . \tag{185}$$

This vector possesses a special property, namely

$$\begin{aligned} \dot{\boldsymbol{A}} &= -\frac{h^2}{r^2}\dot{r}\boldsymbol{e}_r - h\ddot{r}\boldsymbol{e}_\varphi \\ &\quad + \left(\frac{h^2}{r} - GM\right)(\dot{\varphi} + \Omega)\boldsymbol{e}_\varphi + h\dot{r}(\dot{\varphi} + \Omega)\boldsymbol{e}_r \\ &= \left[r^2(\dot{\varphi} + \Omega) - h\right]\frac{h\dot{r}}{r^2}\boldsymbol{e}_r \\ &\quad + \left(-r^2\ddot{r} + \frac{h^2}{r} - GM\right)\frac{h}{r^2}\boldsymbol{e}_\varphi \\ &= \boldsymbol{0} \end{aligned} \tag{186}$$

where,

$$\begin{aligned} \dot{\boldsymbol{e}}_r &= (\dot{\varphi} + \Omega)\boldsymbol{e}_\varphi, \\ \dot{\boldsymbol{e}}_\theta &= 0, \\ \dot{\boldsymbol{e}}_\varphi &= -(\dot{\varphi} + \Omega)\boldsymbol{e}_r, \end{aligned} \tag{187}$$

on $(x\ y)$-plane. Furthermore,

$$\begin{aligned} A^2 &= \boldsymbol{A} \cdot \boldsymbol{A}, \\ \frac{\mathrm{d}}{\mathrm{d}t}A^2 &= 2\dot{\boldsymbol{A}} \cdot \boldsymbol{A} = 0 \end{aligned} \tag{188}$$

since,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}A &= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{A^2} \\ &= -\frac{1}{2\sqrt{A^2}}\frac{\mathrm{d}}{\mathrm{d}t}A^2 \\ &= 0, \end{aligned} \tag{189}$$

thus,

$$\begin{aligned} A^2 &= \left(\frac{h^2}{r} - GM\right)^2 + (h\dot{r})^2 \\ &= (GM)^2 + 2\frac{E}{m}h^2. \end{aligned} \tag{190}$$

where,

$$\dot{r}^2 = 2\frac{E}{m} - \frac{h^2}{r^2} + 2G\frac{M}{r} \tag{191}$$

- Solution Kepler's problem

Using the eccentricity vector $\boldsymbol{A}$, Kepler's problem can be solved.

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{A} = rA\cos(\xi) &= h^2 - GMr \\ r &= \frac{h^2}{GM}\frac{1}{1 + e\cos(\xi)} \end{aligned} \tag{192}$$

where,

[5] a.k.a. Runge-Lenz vector, Laplace-Runge-Lenz vector.

$$e \equiv \frac{A}{GM} = \sqrt{1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2}, \qquad \cos(\xi) \equiv \frac{\boldsymbol{r} \cdot \boldsymbol{A}}{rA} = \boldsymbol{e}_r \cdot \frac{\boldsymbol{A}}{A}. \tag{193}$$

Therefore, noting that $h^2 = r^2(\dot{\varphi} + \Omega)$,

$$r_e = \frac{L}{1 + e\cos(\xi)}, \qquad \omega_e = \frac{h}{L^2}(1 + e\cos(\xi))^2 - \Omega \tag{194}$$

where,

$$L \equiv \frac{h^2}{GM} \tag{195}$$

is obtained. This represents an elliptical orbit, where L is the semi-latus rectum and e is the eccentricity. Here, $\boldsymbol{A}$ is a constant vector, and

$$\boldsymbol{e}_r = \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \\ 0 \end{pmatrix}_{\mathrm{G}}, \quad \varphi(t) = \omega_e t + \varphi_e \tag{196}$$

can be expressed in this form.

For $e = 0$, the orbit is circular,

$$r_e = r_0 = L = \frac{1}{\kappa(\omega_0)}, \qquad \omega_e + \Omega = \omega_0 + \Omega = \frac{h}{L^2} \text{ is const.} \tag{197}$$

which corresponds to the circular motion shown in Equation 164.

For $0 < e < 1$, the orbit is elliptical. Taking φ_e such that at $t = 0$, $\boldsymbol{e}(r)|(t = 0) \cdot \boldsymbol{A}/A = -1$, we have

$$\xi = \omega_e t + \pi, \tag{198}$$

so that

$$r_e = \frac{L}{1 - e\cos(\omega_e t)}. \tag{199}$$

Then, for

$$x_e = r_e\cos(\omega_e t), \quad y_e = r_e\sin(\omega_e t), \tag{200}$$

we find

$$x_e^2 + y_e^2 = r_e^2 = \left(\frac{L}{1 - ex_e\,/\,r_e}\right)^2. \tag{201}$$

Noting that $r_e \geq 0$,

$$r_e = \frac{L}{1 - ex_e\,/\,r_e}, \qquad r_e = L + ex_e. \tag{202}$$

Therefore,

$$\begin{aligned} x_e^2 + y_e^2 &= (L + ex_e)^2 \\ (1 - e^2)x_e^2 - 2eLx_e + y_e^2 &= L^2 \\ x_e^2 - 2\frac{e}{1 - e^2}Lx_e + \frac{y_e^2}{1 - e^2} &= \frac{L^2}{1 - e^2} \\ \left(x_e - \frac{e}{1 - e^2}L\right)^2 + \frac{y_e^2}{1 - e^2} &= \left(\frac{L}{1 - e^2}\right)^2 \\ \frac{\left(x_e - \frac{e}{1-e^2}L\right)^2}{\left(\frac{L}{1-e^2}\right)^2} + \frac{y_e^2}{\left(\frac{L}{\sqrt{1-e^2}}\right)^2} &= 1, \end{aligned} \tag{203}$$

hence the standard form of the ellipse is

$$\frac{(x_e - f_e)^2}{a^2} + \frac{(y_e)^2}{b^2} = 1, \tag{204}$$

where

$$a \equiv \frac{L}{1 - e^2}, \quad b \equiv \frac{L}{\sqrt{1 - e^2}}, \qquad f_e \equiv \frac{e}{1 - e^2}L = \sqrt{a^2 - b^2}. \tag{205}$$

Here a, b are the semi-major and semi-minor axes ($0 < b \leq a$), and $x_e = f_e$ corresponds to one focus of the ellipse (the other being at $x_e = -f_e$). Note that

$$L = \frac{b^2}{a}, \quad e = \sqrt{1 - \left(\frac{b}{a}\right)^2}. \tag{206}$$

For $e = 1$, the trajectory is parabolic. At $t = 0$, taking $\boldsymbol{e}(r)|(t = 0) \cdot \boldsymbol{A}/A = 1$, we have

$$\xi = \omega_e t \tag{207}$$

and thus

$$r_e = \frac{L}{1 + \cos(\omega_e t)} = \frac{L}{1 + x_e/r_e} \tag{208}$$

so that

$$r_e = L - x_e \tag{209}$$

and the parabolic trajectory is

$$\begin{aligned} x_e^2 + y_e^2 &= (L - x_e)^2 \\ y_e^2 &= 2x_e^2 - 2Lx_e + L^2 \\ y_e &= \pm\sqrt{2\left(x_e - \frac{L}{2}\right)^2 + \frac{L^2}{2}} \end{aligned} \tag{210}$$

is obtained.

Finally, for $e > 1$, the trajectory becomes hyperbolic[6]. Taking

$$\xi = \omega_e t \tag{211}$$

so that $\boldsymbol{e}(r)|(t=0) \cdot \boldsymbol{A}/A = 1$, and noting $r_e, L \geq 0$,

$$\begin{aligned} r_e^2 &= \left(\frac{L}{1+e\cos(\omega_e t)}\right)^2 = \left(\frac{L}{1+ex_e/r_e}\right)^2 \\ (r_e+ex_e)^2 &= L^2 \\ r_e &= L - ex_e. \end{aligned} \tag{212}$$

On the other hand, since $x_e^2 + y_e^2 = r_e^2$,

$$\begin{aligned} x_e^2 + y_e^2 &= (L - ex_e)^2 \\ -(e^2-1)x_e^2 + 2eLx_e + y_e^2 &= L^2 \\ x_e^2 - 2\frac{e}{e^2-1}Lx_e - \frac{y_e^2}{e^2-1} &= -\frac{L^2}{e^2-1} \\ \left(x_e - \frac{e}{e^2-1}L\right)^2 - \frac{y_e^2}{e^2-1} &= \left(\frac{L}{e^2-1}\right)^2 \\ \frac{\left(x_e - \frac{e}{e^2-1}L\right)^2}{\left(\frac{L}{e^2-1}\right)^2} - \frac{y_e^2}{\left(\frac{L}{\sqrt{e^2-1}}\right)^2} &= 1, \end{aligned} \tag{213}$$

so that

$$\frac{(x_e - f_e)^2}{a^2} - \frac{(y_e)^2}{b^2} = 1 \tag{214}$$

where

$$\begin{aligned} a &\equiv \frac{L}{e^2-1}, \quad b \equiv \frac{L}{\sqrt{e^2-1}}, \\ f_e &\equiv \frac{e}{e^2-1}L. \end{aligned} \tag{215}$$

Similarly, it holds

$$L = \frac{b^2}{a}, \quad e = \sqrt{1 + \left(\frac{b}{a}\right)^2}. \tag{216}$$

Quadrupole effect on galactic plane

Next, let us consider the force due to the quadrupole moment. On the galactic plane (the xy-plane), the force arising from the quadrupole moment is concisely obtained from Equation 155 as

$$\begin{aligned} f^r &= -\frac{3}{5}M\left(q^{xx}\cos^2(\varphi) + q^{yy}\sin^2(\varphi)\right) \\ f^\theta &= 0 \\ f^\varphi &= -\frac{1}{5}Mq^{xx}\sin(2\varphi), \end{aligned} \tag{217}$$

where

$$\begin{aligned} q^{xx} &= 2a^2 - b^2 - c^2, \\ q^{yy} &= 2b^2 - a^2 - c^2. \end{aligned} \tag{218}$$

Substituting these relations yields

$$\begin{aligned} f^r(\varphi) &= -\frac{3}{5}M[3(a^2\cos^2(\varphi) + b^2\sin^2(\varphi)) \\ &\qquad -(a^2+b^2+c^2)], \\ f^\theta(\varphi) &= 0, \\ f^\varphi(\varphi) &= -\frac{1}{5}M(2a^2-b^2-c^2)\sin(2\varphi). \end{aligned} \tag{219}$$

- Equation of motion with the quadrupole force

The motion considered here is governed by the equation of motion Equation 170, with the quadrupole term incorporated:

$$\begin{aligned} & m\begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 0 \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) \end{pmatrix}_{\text{polar}} \\ &= 2m\Omega\begin{pmatrix} r\dot{\varphi} \\ 0 \\ -\dot{r} \end{pmatrix}_{\text{polar}} + m\Omega^2 r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} \\ &\quad -G\frac{m}{r^2}\begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}_{\text{polar}} \\ &\quad -G\frac{m}{2r^4}\frac{M}{5}\begin{pmatrix} 3(q^{xx}\cos^2(\varphi) + q^{yy}\sin^2(\varphi)) \\ 0 \\ q^{xx}\sin(2\varphi) \end{pmatrix}_{\text{polar}}. \end{aligned} \tag{220}$$

This equation of motion is derived from the Euler–Lagrange equation associated with the Lagrangian

$$L(\boldsymbol{r}, \boldsymbol{v}) = \frac{1}{2}m|\boldsymbol{v}(\boldsymbol{r}, \dot{\boldsymbol{r}})|^2 - m\Phi(\boldsymbol{r}). \tag{221}$$

Energy conservation, $\dot{E} = 0$, holds with

$$E \equiv T + m\Phi, \text{ where } \Phi = \Phi^{(m)} + \Phi^{(q)} \tag{222}$$

and from Equation 99 we obtain

[6]Such orbits are also utilized in satellite flybys.

$$
\begin{aligned}
\dot{E} &= \ddot{\boldsymbol{r}} \cdot \frac{\partial T}{\partial \dot{\boldsymbol{r}}} + \dot{\boldsymbol{r}} \cdot \frac{\partial E}{\partial \boldsymbol{r}} \\
&= \ddot{\boldsymbol{r}} \cdot (m\dot{\boldsymbol{r}}) + \dot{\boldsymbol{r}} \cdot \left(\frac{\partial T}{\partial \boldsymbol{r}} + m\frac{\partial \Phi}{\partial \boldsymbol{r}}\right) \\
&= \dot{\boldsymbol{r}} \cdot \left(m\ddot{\boldsymbol{r}} + \frac{\partial T}{\partial \boldsymbol{r}} + m\frac{\partial \Phi}{\partial \boldsymbol{r}}\right) \qquad = 0,
\end{aligned} \tag{223}
$$

On the other hand, from the φ-direction equation of motion, the angular momentum $h \equiv r^2(\dot{\varphi} + \Omega)$ satisfies

$$
\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{G}{2r^4}\frac{M}{5}q^{xx}\sin(2\varphi) \tag{224}
$$

which shows that the quadrupole term breaks the conservation of angular momentum $h, \boldsymbol{h}$. The same applies to the eccentricity vector $\boldsymbol{A}$, whose conservation is also violated [11].

Accordingly, let us explicitly summarize the equations of motion in each direction:

- r-direction

$$
\begin{aligned}
\ddot{r} = {} & \frac{h^2}{r^3} - G\frac{M}{r^2} \\
& -GM\frac{3}{10r^4}\left(q^{xx}\cos^2(\varphi) + q^{yy}\sin^2(\varphi)\right)
\end{aligned} \tag{225}
$$

- φ-direction

$$
\dot{h} = -G\frac{M}{10r^4}q^{xx}\sin(2\varphi) \tag{226}
$$

Elevational motion

We now extend our discussion to a more general case that includes motion in the elevational (θ) direction.

Rearranging Equation 113 and Equation 155, the equations of motion can be expressed as follows:

- r-direction

$$
\begin{aligned}
\ddot{r} = {} & \frac{1}{r^3}\left(h_\varphi^2 + h_\theta^2\right) \\
& -G\frac{M}{r^2} \\
& -GM\frac{3}{10r^4}\big(q^{xx}\cos^2(\theta)\cos^2(\varphi) \\
& \qquad +q^{yy}\cos^2(\theta)\sin^2(\varphi) \\
& \qquad +q^{zz}\sin^2(\theta)\big)
\end{aligned} \tag{227}
$$

where

$$
h_\varphi \equiv r^2(\dot{\varphi} + \Omega)\cos(\theta), \quad h_\theta \equiv r^2\dot{\theta} \tag{228}
$$

- θ-direction

$$
\begin{aligned}
\dot{h_\theta} = {} & -\frac{h_\varphi^2}{r^2}\tan(\theta) \\
& -G\frac{M}{10r^3}(q^{xx} - q^{yy})\sin^2(\varphi)
\end{aligned} \tag{229}
$$

- φ-direction

$$
\begin{aligned}
\dot{h_\varphi} = {} & \frac{h_\theta}{r^2}h_\varphi\tan(\theta) \\
& -G\frac{M}{5r^3}\cos(\varphi)(q^{xx}\cos(\theta)\sin(\varphi) - q^{yy}\sin(\theta))
\end{aligned} \tag{230}
$$

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Contributions

imash performed the modeling and analysis. All authors designed the work and wrote the paper.

Conflicts of Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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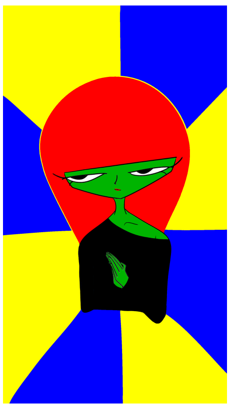
Nishiko



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Noham Burger

Noham Burger.

Anti(=反 /HAM)Burger who went to the ubiquitous consciousness of mankind.

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HARPA: Hamburger Advanced Research Projects Agency

The Hamburger Advanced Research Projects Agency is an independent institution, unaffiliated with any university, dedicated to flexibly exploring new perspectives in scholarship that are often constrained by conventional structures. It operates with a high degree of freedom. The DPK, Digital Potato Kingdom, is a "kingdom" in name only: like a hamburger in which diverse layers overlap yet ultimately converge between the same buns, it has no bureaucracy or hierarchy. We are continuously seeking like-minded collaborators.

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